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THE STRUCTURE OF BOUNDED BILINEAR FORMS ON PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. Let A_1 and A_2 be C^*-algebras and $B: A_1 \times A_2 \to \mathbf{C}$ be a bounded bilinear form. It is proved that there exist a Hilbert space H, two Jordan morphisms $\mu_i: A_i \to L(H)$, $i = 1,2$, and two vectors $\xi_1, \xi_2 \in H$ such that

$$B(x,y) = (\mu_1(x)\xi_1 \mid \mu_2(y^*)\xi_2) \quad \text{for all } x \in A_1,\ y \in A_2.$$

The proof depends on the Grothendieck-Pisier-Haagerup inequality and Halmos's unitary dilation theorem. An extemely elementary proof of the latter is given.

1. Introduction. In [**8**, Remark 5.3(a)] it was observed that some arguments in the proof of [**8**, Theorem 5.2] yield (essentially, cf. Remark 2.2) the following representation theorem for bounded bilinear forms on the Cartesian product of two C^*-algebras.

1.1. LEMMA. *Let A_1 and A_2 be two C^*-algebras and $B: A_1 \times A_2 \to \mathbf{C}$ be a bounded bilinear form. Then there exist four Hilbert spaces K_i', K_i'', $i = 1,2$, cyclic $*$-representations $\pi_i': A_i \to L(K_i')$ with unit cyclic vectors $\xi_1' \in K_i'$, cyclic $*$-antirepresentations $\pi_i'': A_i \to L(K_i'')$ with unit cyclic vectors $\xi_i'' \in K_i''$ and a bounded linear map $T: K_1' \oplus K_1'' \to K_2' \oplus K_2''$ with $\|T\| \le \|B\|$ such that for all $x \in A_1$, $y \in A_2$*

$$B(x,y) = (T(\pi_1'(x)\xi_1', \pi_1''(x)\xi_1'') | (\pi_2'(y^*)\xi_2', \pi_2''(y^*)\xi_2''))_{K_2' \oplus K_2''}. \tag{1}$$

Here and elsewhere, for any Hilbert space H, $(\cdot|\cdot)_H$ or $(\cdot|\cdot)$ denotes the inner product and $L(H)$ the space of bounded linear operators on H. If K is a closed subspace of H, P_K is the orthogonal projection of H onto K.

The aim of this note is to point out that a representation in terms of Jordan morphisms can be obtained even without the help of the operator T. In the formulation of Theorem 2.1 the term "Jordan morphism" is only used for euphony; Jordan morphisms are known to be precisely the direct sums of $*$-representations and $*$-antirepresentations [**12**], and the proof actually yields such direct sums.

In the appendix we spell out for completeness the details of a proof of Lemma 1.1, since the proof is short and our point of view differs somewhat from that of [**8**]. We also find it worthwhile to communicate an utterly short and elementary proof of the Halmos dilation theorem used in the proof of Theorem 2.1.

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2. The main result.

2.1. THEOREM. *Let A_1 and A_2 be C^*-algebras and $B\colon A_1 \times A_2 \to \mathbf{C}$ be a bounded bilinear form. There exist a Hilbert space H, two Jordan morphisms $\mu_i\colon A_i \to L(H)$, $i = 1, 2$, and two vectors $\xi_1, \xi_2 \in H$ such that*

$$B(x,y) = (\mu_1(x)\xi_1|\mu_2(y^*)\xi_2) \quad \textit{for all } x \in A_1, y \in A_2.$$

PROOF. We use Lemma 1.1 and its notation, and write $K = K_1 \oplus K_2$ where $K_i = K_i' \oplus K_i''$, $i = 1, 2$. We modify a technique used in the proof of Theorem 2.4 in [**4**]. Define $\tilde{T}\colon K \to K$ by the formula $\tilde{T}(u,v) = (0, Tu)$. We may assume that $\|B\| \le 1$, so that $\|\tilde{T}\| \le 1$. Thus $\tilde{T}$ has a unitary dilation, i.e., there is a Hilbert space H containing K as a subspace such that $T = P_K U|K$ for some unitary operator $U\colon H \to H$ (see §3 for a proof and references). Write H in the form $H = K_1 \oplus K_2 \oplus K^\perp$, denote $\pi_i = \pi_i' \oplus \pi_i''\colon A_i \to L(K_i)$ for $i = 1, 2$, and define $\mu_1(x)(\varsigma_1, \varsigma_2, \varsigma_3) = (\pi_1(x)\varsigma_1, 0, 0)$ for $x \in A_1$, $\varsigma_1 \in K_1$, $\varsigma_2 \in K_2$, $\varsigma_3 \in K^\perp$, and $\mu_2(y) = U^*(0 \oplus \pi_2(y) \oplus 0)U$ for $y \in A_2$. Moreover, denote $\xi_1 = ((\xi_1', \xi_1''), 0, 0)$ and $\xi_2 = U^*(0, (\xi_2', \xi_2''), 0)$. A direct calculation using (1) in Lemma 1.1 now gives

$$\begin{aligned}
&(\mu_1(x)\xi_1|\mu_2(y^*)\xi_2)_H \\
&\quad = (U(\pi_1(x)(\xi_1', \xi_1''), 0, 0)|(0, \pi_2(y^*)(\xi_2', \xi_2''), 0))_H \\
&\quad = (\tilde{T}(\pi_1(x)(\xi_1', \xi_1''), 0)|(0, \pi_2(y^*)(\xi_2', \xi_2'')))_K \\
&\quad = (T\pi_1(x)(\xi_1', \xi_1'')|\pi_2(y^*)(\xi_2', \xi_2''))_{K_2} = B(x,y). \quad \square
\end{aligned}$$

2.2. REMARK. Applying Theorem 2.1 to $\tilde{B}\colon A_2 \times A_1 \to \mathbf{C}$ defined by $\tilde{B}(y,x) = B(x,y)$ we see that B can also be represented in the form $B(x,y) = (\mu_1(x)\mu_2(y)\xi_1|\xi_2)$, where $\mu_i\colon A_i \to L(H)$ are Jordan morphisms for some Hilbert space H, and $\xi_i \in H$ for $i = 1, 2$. Similarly, the technique of the above proof shows that in [**3**, Theorem 2.1, condition (3)], one may take $H = K$ and leave the operator T out. Theorem 2.1 of the present paper was announded at the UCLA Functional Analysis seminar October 8, 1986; I am indebted to E. G. Effros for bringing a preprint of [**3**] in that connection to my attention.

3. Appendix: alternate proofs of essentially known results. The unitary dilation result in [**7**, Problem 177(a)] due to Halmos [**6**] is contained in part (b) of the following proposition. Unlike [**6**, **7**] or the proof of (a) in [**2**], the proof below does not depend on the existence of the square root of a positive operator. The proof of (a) may actually be seen as flowing from a comparison of [**2**] with a more traditional approach (see e.g. [**1**, **9**–**11**]) to the kind of problem treated there.

3.1. PROPOSITION. *Let H_1 and H_2 be Hilbert spaces and $T\colon H_1 \to H_2$ a linear contraction.*

(a) *There exist a Hilbert space K_2 containing H_2 as a subspace, and an isometric linear map $V\colon H_1 \to K_2$ such that $T = P_{H_2}V$.*

(b) *There exist Hilbert spaces K_1 and K_2 and an isometric linear surjection $U\colon K_1 \to K_2$ such that K_i contains H_i as a subspace, $i = 1, 2$, and $T = P_{H_2}U|H_1$.*

PROOF. (a) Denote $h(x,y) = (x|y) - (Tx|Ty)$ for $x, y \in H_1$. Then $h\colon H_1 \times H_1 \to \mathbf{C}$ is sesquilinear, and positive since $\|T\| \le 1$. Denote $N = \{x \in H_1 | h(x,x) = 0\}$, and complete H_1/N with the inner product $(x + N|y + N) = h(x,y)$ to a

Hilbert space K. Finally define $K_2 = K \oplus H_2$, and $Vx = (x + N, Tx) \in K_2$ for $x \in H_1$. Interpreting H_2 as a subspace of K_2, we have $T = P_{H_2}V$, and $\|Vx\|^2 = \|x + N\|^2 + \|Tx\|^2 = \|x\|^2$.

(b) Continuing from (a), denote $K_1 = H_1 \oplus (K_2 \ominus V(H_1))$, and define $U(x, y) = Vx + y$ for $x \in H_1$, $y \in K_2 \oplus V(H_1)$. □

PROOF OF LEMMA 1.1. We may assume that $\|B\| \leq 1$. Denote $B_0(x, y) = B(x, y^*)$, so that B_0 is a bounded sesquilinear form with $\|B_0\| \leq 1$. Using Haagerup's general version [**5**, Theorem 1.1] of Pisier's Grothendieck type inequality we get four states $\varphi_i, \psi_i \colon A_i \to \mathbf{C}$, $i = 1, 2$, such that

$$|B_0(x, y)| \leq [\varphi_1(x^*x) + \psi_1(xx^*)]^{1/2}[\varphi_2(y^*y) + \psi_2(yy^*)]^{1/2}$$

for all $x \in A_1$, $y \in A_2$. (In a similar estimate for B exchange the roles of the last two states.) Denoting $h_i(u, v) = \varphi_i(v^*u) + \psi_i(uv^*)$ and $N_i = \{u \in A_i | h_i(u, u) = 0\}$, we get two inner product spaces A_i/N_i with the inner products $(u + N_i | v + N_i) = h_i(u, v)$. Completing these to Hilbert spaces K_i we obtain a well-defined bounded sesquilinear form $\tilde{B}_0 \colon K_1 \times K_2 \to \mathbf{C}$ with $\|\tilde{B}_0\| \leq 1$ and $\tilde{B}_0(x + N_1, y + N_2) = B_0(x, y)$, $x \in A_1$, $y \in A_2$. Thus there is a bounded linear operator $\tilde{T} \colon K_1 \to K_2$ such that $\tilde{B}_0(w, z) = (\tilde{T}w | z)$, $w \in K_1$, $z \in K_2$. On the other hand, applying the GNS-construction [**13**, Theorem 9.14] to A_i and φ_i, and to the opposite C^*-algebra (i.e., otherwise the same as A_i but equipped with the product $u \cdot v = vu$ where vu is the product of A_i) and ψ_i we see that $\varphi_i(u) = (\pi_i'(u)\xi_i' | \xi_i')$ and $\psi_i(u) = (\pi_i''(u)\xi_i'' | \xi_i'')$ for a cyclic $*$-representation $\{\pi_i', K_i', \xi_i'\}$ and a cyclic $*$-antirepresentation $\{\pi_i'', K_i'', \xi_i''\}$ of A_i. A straightforward calculation shows that $(u + N_i | v + N_i)$ is the same as the inner product of $(\pi_i'(u)\xi_i', \pi_i''(u)\xi_i'')$ and $(\pi_i'(v)\xi_i', \pi_i''(v)\xi_i'')$ in $K_i' \oplus K_i''$, which implies that we get for each $i = 1, 2$ a well-defined isometric linear map $V_i \colon K_i \to K_i' \oplus K_i''$ satisfying $V_i(u + N_i) = (\pi_i'(u)\xi_i', \pi_i''(u)\xi_i'')$ for $u \in A_i$. It is now easy to verify that the choice $T = V_2\tilde{T}V_1^*$ will work. □

REFERENCES

1. J. L. Abreu, *A note on harmonizable and stationary sequences*, Bol. Soc. Mat. Mexicana **15** (1970), 48–51.
2. S. D. Chatterji, *Orthogonally scattered dilation of Hilbert space valued set functions*, Measure Theory (Proc. Conf. Oberwolfach, 1981), Lecture Notes in Math., vol. 945, Springer-Verlag, Berlin and New York, 1982, pp. 269–281.
3. E. G. Effros and A. Kishimoto, *Module maps and Hochschild-Johnson cohomology*, Indiana Univ. J. Math. **36** (1987), 257–276.
4. C. C. Graham and B. M. Schreiber, *Bimeasure algebras on LCA groups*, Pacific J. Math. **115** (1984), 91–127.
5. U. Haagerup, *The Grothendieck inequality for bilinear forms on C^*-algebras*, Advances in Math. **56** (1985), 93–116.
6. P. R. Halmos, *Normal dilations and extensions of operators*, Summa Brasil. Math. **2** (1950), 125–134.
7. ——, *A Hilbert space problem book*, Van Nostrand, Toronto, Princeton, N.J., and London, 1967.
8. S. Kaijser and A. M. Sinclair, *Projective tensor products of C^*-algebras*, Math. Scand. **55** (1984), 161–187.
9. A. G. Miamee and H. Salehi, *Harmonizability, V-boundedness and stationary dilation of stochastic processes*, Indiana Univ. Math. J. **27** (1978), 37–50.

10. H. Niemi, *On stationary dilations and the linear prediction of certain stochastic processes*, Soc. Sci. Fenn. Comment. Phys.-Math. **45** (1975), 111–130.
11. ——, *On orthogonally scattered dilations of bounded vector measures*, Ann. Acad. Sci. Fenn. AI Math. **3** (1977), 43–52.
12. E. Størmer, *On the Jordan structure of C^*-algebras*, Trans. Amer. Math. Soc. **120** (1965), 438–447.
13. M. Takesaki, *Theory of operator algebras*. I, Springer-Verlag, New York, Heidelberg and Berlin, 1979.

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