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A NOTE ON MEASURABILITY AND ALMOST CONTINUITY

MAXIM R. BURKE AND DAVID H. FREMLIN

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ABSTRACT. We prove that it is consistent with ZFC that there exist a measurable function f: $[0,1] \to \omega_1$ which is not almost continuous.

DEFINITION. If Y is a topological space, a function f: $[0,1] \to Y$ is *measurable* if $f^{-1}(G)$ is Lebesgue measurable for each open $G \subset Y$, and is *almost continuous* if for every $\varepsilon > 0$, there is a measurable set E with $\mu E > 1 - \varepsilon$, and $f|_E$ continuous.

In [**2**, Theorem 2B], Fremlin proves

THEOREM 1. *If X is a Radon measure space and Y is a metric space, then a function f: $X \to Y$ is measurable iff it is almost continuous.*

(See [**2**] for the definition of a Radon measure space. All we need to know here is that $[0,1]$ with Lebesgue measure is such a space.)

Assuming GCH there is the following result for nonmetrizable spaces Y.

THEOREM 2 [**2**, THEOREM 3G]. (GCH) *If X is Radon and ς is an ordinal (with the order topology), then every function f: $X \to \varsigma$ is measurable iff it is almost continuous.*

What happens to Theorem 2 if GCH is dropped is left open in [**2**] even when $X = [0,1]$, $\varsigma = \omega_1$. In this paper we show that Theorem 2 can fail in this case:

THEOREM 3. *If $\mathfrak{c} = 2^{\aleph_0} = 2^{\aleph_1}$ and there is a collection $\{N_\xi: \xi > \mathfrak{c}\}$ of null (= Lebesgue negligible) subsets of $[0,1]$ such that $\forall L \in [\mathfrak{c}]^{\omega_1}$, $\bigcup_{\xi \in L} N_\xi = [0,1]$, then there is a function f: $[0,1] \to \omega_1$ which is measurable but not almost continuous.*

REMARK. The hypothesis of Theorem 3 holds in the model obtained by adding $2^{\aleph_1}$ Cohen reals to a model of ZFC. (The collection $\{N_\xi: \xi < \mathfrak{c}\}$ is obtained by taking the translates of a fixed null comeager set H by the elements of a Luzin set of size $\mathfrak{c}$. See [**1**].)

PROOF OF THEOREM 3. Let $\langle C_\xi: \xi < \mathfrak{c}\rangle$ enumerate the club subsets of ω_1. We define a collection $\{E_\eta : \eta < \omega_1\}$ of null sets satisfying

(i) $\forall \eta < \omega_1$ $(E_\eta \subset N_\eta)$,

(ii) $\forall \xi < \mathfrak{c}$ $(\bigcup_{\eta \in C_\xi} E_\eta \supset [0,1] \setminus N_\xi)$,

(iii) $\forall \eta, \eta' < \omega_1$, $(\eta \neq \eta' \Rightarrow E_\eta \cap E_{\eta'} = \emptyset)$.

Given $x \in [0,1]$, let $\mathcal{A}(x) = \{\xi < \mathfrak{c}: x \notin N_\xi\} \cup \{0\}$. This is a nonvoid countable set. Let $\eta(x)$ be the least member of $\bigcap_{\xi \in \mathcal{A}(x)} C_\xi$ s.t. $x \in N_{\eta(x)}$. Define $E_\eta = \{x \in [0,1]: \eta = \eta(x)\}$. It is easy to check (i), (ii), and (iii).

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Now define $f\colon [0,1] \to \omega_1$ by $f^{-1}(\{\eta\}) = E_\eta$. By (ii), $f^{-1}(C)$ is measurable when C is closed, and thus f is measurable. But f is not almost continuous since if K is compact, $\mu K > 0$, and $f|_K$ is continuous then $f(K)$ is a compact subset of ω_1; thus $f(K)$ is countable and so K is a countable union of null sets, contradicting $\mu K > 0$.

This completes the proof.

REMARK. J. Cichoń has pointed out that the function f can be constructed so that $f^{-1}(A)$ is null iff A is nonstationary: Let $\{B_\gamma \colon \gamma < \mathfrak{c}\}$ be a decomposition of $[0,1]$ into disjoint Bernstein sets, let $\langle S_\gamma \colon \gamma < \mathfrak{c}\rangle$ enumerate the stationary subsets of ω_1. Then in the proof above choose $\eta(x)$ from $\bigcap_{\xi \in \mathcal{A}(x)} C_\xi \cap S_\gamma$ where γ is chosen so that $x \in B_\gamma$.

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, M5S 1A1, CANADA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ESSEX, COLCHESTER CO4 3SQ, ENGLAND