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A NOTE ON PSEUDOCONVEXITY AND PROPER HOLOMORPHIC MAPPINGS

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ABSTRACT. In this paper we discuss some connections between proper holomorphic mappings between domains in $\mathbf{C}^n$ and the boundary behaviors of certain canonical invariant metrics. A compactness theorem has been proved. This generalizes slightly an earlier result proved by the second author.

Introduction. A continuous mapping $f\colon X_1 \to X_2$ between two topological spaces is called proper if $f^{-1}(K) \subset X_1$ is compact whenever $K \subset X_2$ is compact. Proper holomorphic mappings between analytic spaces stand out for their beauty and simplicity. For instance, if $g\colon D_1 \to D_2$ is a proper holomorphic mapping between two bounded domains in $\mathbf{C}^n$, a theorem of Remmert says that (D_1, g, D_2) is a finite branching cover. The branching locus in D_1 is described by $\{z \in D_1 \mid \det(dg(z)) = 0\}$. For the past ten years, there has been a great amount of activity in characterizing the proper holomorphic mappings between pseudoconvex domains. It has been known for a long time that there are numerous proper holomorphic maps between unit disks in $\mathbf{C}^1$. The simplest example is $g\colon \Delta = \{z \in \mathbf{C}^1 \mid |z| < 1\} \to \Delta$, $g(z) = z^n$, where n is any positive integer. Nevertheless, such a phenomenon is no longer true in higher-dimensional cases. H. Alexander was able to verify the following interesting fact.

THEOREM 1 [**1**]. *Let $B_n = \{(z_1, z_2, \ldots, z_n) | \sum_{i=1}^n |z_i|^2 < 1\}$ be the unit ball in $\mathbf{C}^n$, $n \geq 2$. Suppose $f\colon B_n \to B_n$ is a proper holomorphic mapping. Then f must be a biholomorphism.*

The following result due to S. Pinčuk is an extension of Alexander's theorem.

THEOREM 2 [**5**]. *Let D_1 and D_2 be two strongly pseudoconvex bounded domains with smooth boundaries in $\mathbf{C}^n$, $n \geq 2$. Suppose $f\colon D_1 \to D_2$ is a proper holomorphic mapping. Then f is a covering.*

In [**7**] the second author proved the following result concerning biholomorphic groups of strongly pseudoconvex domains.

THEOREM 3 [**7**]. *Let D be a strongly pseudoconvex bounded domain with smooth boundary in $\mathbf{C}^n$. Then* $\operatorname{Aut}(D)$ *is noncompact iff D is biholomorphic to B_n, $n = \dim_{\mathbf{C}} D$.*

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In view of a lot of recent attention on the topic of proper holomorphic mappings, the authors feel that it might be worthwhile to point out the following startling fact which generalizes Theorem 3.

THEOREM 4. *Let D_1 and D_2 be two strongly pseudoconvex bounded domains with smooth boundaries in $\mathbb{C}^n$, $n \geq 2$. Then $P(D_1, D_2)$ is noncompact iff both D_1 and D_2 are biholomorphic to B_n, where $P(D_1, D_2)$ denotes the set of all proper holomorphic mappings between D_1 and D_2.*

Pinčuk's Theorem 2 says that proper holomorphic mappings between strongly pseudoconvex domains are unbranching. It follows that Theorem 4 is an immediate consequence of the local version stated next, which is the principal result of this note.

THEOREM 5. *Let D_1 and D_2 be bounded domains in $\mathbb{C}^n$. We denote $P_0(D_1, D_2)$ as the set of all unbranching proper holomorphic maps from D_1 to D_2. Suppose the following two conditions are fulfilled.*

(1) *There is a strongly pseudoconvex boundary point $p \in \partial D_2$.*

(2) *There exists a point $x \in D_1$ and a sequence $\{f_j\} \subseteq P_0(D_1, D_2)$ such that $\{f_j(x)\}$ converges to p.*

Then both D_1 and D_2 are biholomorphic to B_n.

(A) Some preliminaries and related results. Let M be a complex manifold of dimension n, $x \in M$, and k an integer between one and n.

DEFINITION. The Eisenman differential k-measure on M is a function E^k_M: $\bigwedge^k T(M) \to \mathbf{R}$ such that for all $(x, v) \in \bigwedge^k T_x(M)$,

$$E^k_M(x,v) = \inf \left\{ R^{-2^k} \mid \text{there exists a holomorphic map } f\colon B_k(R) \to M \text{ such that } f(0) = x \text{ and } df_0\left(\frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_2} \wedge \cdots \wedge \frac{\partial}{\partial w_k}(0)\right) = v \right\},$$

where $B_k(R) = \{w = (w_1, w_2, \ldots, w_k) \in C^k \mid \sum_{i=1}^k |w_i|^2 < R\}$.

When $k = 1$, it is called a Kobayashi-Royden differential metric [**6**], denoted $K_M = k\sqrt{E^1_M}$. As $k = n$, it is a volume form, denoted by $E^n_M = |E^n_M|\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|E^n_M|$ is a function on M.

On the other hand, the Carathéodory differential k-measure C^k_M is defined as follows.

DEFINITION. $C^k_M\colon \bigwedge^k T_x(M) \to \mathbf{R}$, $(x, v) \in \bigwedge_k T_x(M)$, $C^k_M(x,v) = \sup\{1/R^{2^k} \mid$ there exists a holomorphic mapping $f\colon M \to B_k(R)$ such that $f(x) = 0$, $df_x(v) = \partial/\partial w_1 \wedge \cdots \wedge \partial/\partial w_k(0)\}$.

When $k = 1$, it is called a Carathéodory-Reiffen differential metric, denoted by $C_M = \sqrt{C^1_M}$. As $k = n$, it is a volume form $C^n_M = |C^n_M|\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|C^n_M|$ is a function on M.

One can also define E^k_M and C^k_M relative to a polydisc instead of a ball. They are different measures, but enjoy similar properties. In the sequel, we shall use I^k_M to represent either E^k_M or C^k_M.

The following theorem follows almost immediately from the definitions [**4**].

THEOREM (a). (1) *$E^k_M \geq C^k_M$ on any complex manifold M.*

(2) *Let $f\colon M_1 \to M_2$ be a holomorphic mapping between complex manifolds M_1 and M_2. Then one has $I^k_{M_1} \geq f^*(I^k_{M_2})$, a measure-decreasing property under f.*

(3) *Let X be a domain of a complex manifold Y. Then $I^k_X \geq I^k_Y$, a monotone property, holds.*

(4) *Any biholomorphism f of a complex manifold X is measure-preserving relative to I^k_X, that is, $I^k_X = f^*(I^k_X)$.*

(5) *Let $\widetilde{M}$ be a covering of a complex manifold M. Denote $\pi\colon \widetilde{M} \to M$ as the covering projection. Then $E^k_{\widetilde{M}} = \pi^*(E^k_M)$.*

THEOREM (b) [**3**, **8**]. *Let D be a bounded domain in $\mathbb{C}^n$ with a strongly pseudoconvex boundary point $p \in \partial D$. We denote $\widetilde{D} = V \cap D$, where $p \in V$ is a sufficiently small ball in $\mathbb{C}^n$. Then the following is true: $|E^n_{\widetilde{D}}(z)|/|C^n_{\widetilde{D}}(z)|$ approaches one as $z \to p$.*

In [**7**], the next theorem was proved for the special case where D is completely hyperbolic. Actually, a similar proof can yield a slightly more general statement as follows.

THEOREM (c) [**7**]. *Let D be a bounded domain in $\mathbb{C}^n$. Suppose that there is one point $x \in D$ such that $|E^n_D(x)| = |C^n_D(x)|$. Then D is biholomorphic to the euclidean ball.*

THEOREM (d) [**2**] (CARTAN'S FIXED POINT THEOREM). *Let (X, ds^2) be a simply-connected complete Riemannian manifold with nonpositive sectional curvature. Suppose G is a compact Lie group acting on X as isometries. Then G has a fixed point.*

In particular, any finite group H acting on X isometrically must fix at least one point.

THEOREM (e). *Let D_1 and D_2 be bounded domains in $\mathbb{C}^n$. Suppose that*

(1) *there is a strongly pseudoconvex point $p \in \partial D_2$;*

(2) *one can find $x \in D_1$ and a sequence of holomorphic mappings $\{f_j\} \subset \mathrm{Hol}(D_1, D_2)$ such that $\{f_j(x)\} \to p$.*

Then there exists a subsequence of $\{f_j\}$, denoted by the same notation $\{f_j\}$, satisfying the property: For any compact set $K \subset D_1$ and any open set $\widetilde{D} = V \cap D_2$, where $p \in V$ is an open set in $\mathbb{C}^n$, there is a j_0 in such a way that $f_j(K) \subset \widetilde{D}$ for all $j \geq j_0$.

PROOF. Since $\{f_j(x)\} \to p$, by normal family argument one can find a subsequence of $\{f_j\}$ converging on compacta to a holomorphic mapping $f\colon D_1 \to \mathbb{C}^n$ so that $f(x) = p$ and $f(D_1) \subseteq \partial D_2$. By assumption, ∂D_2 is strongly pseudoconvex at p and it contains no complex analytic variety of positive dimension through p. This implies f is a constant mapping which brings the whole D onto a single point. Our claim in Theorem (e) should now be clear.

(B) Proof. Let us assume $|E^n_{D_1}(x)| = |C^n_{D_1}(x)|$ for the given point x in D_1. By Theorem (c), this implies that D_1 must be biholomorphic to B_n. If the order of the covering $f_j\colon B_n = D_1 \to D_2$ is greater than one, this would contradict Cartan's fixed point theorem (Theorem (c)) because the Bergman metric on B_n has negative sectional curvature and it is invariant under biholomorphisms. Thus D_2 is also biholomorphic to B_n. Therefore, the whole proof depends on the following assertion.

Claim. $|E^n_{D_1}(x)| = |C^n_{D_1}(x)|$.

PROOF. For each j, $f_j\colon D_1 \to D_2$ is a covering. From Theorem (a)(5) we have

$$E^n_{D_1}(x,v) = E^n_{D_2}(x_j, df_j(v)),$$

where $x_j = f_j(x)$ and (x,v) is a nonzero n-vector at x. Let $(D_1)_k$ be an increasing sequence of domains such that $\bigcup_{k=1}^{\infty}(D_1)_k = D_1$, $x \in (D_1)_k$ for each k, and $(D_1)_k \subset\subset (D_1)_{k+1}$. For each j, denote $(D_2)^j_k = f_j(D_1)_k$. For a fixed k, we obtain by Theorem (a)(2)(3) the inequalities

$$C^n_{(D_1)_k}(x,v) \geq C^n_{(D_1)^j_k}(x_j, df_j(v)) \geq C^n_{\widetilde{D}}(x_j, df_j(v)).$$

The last inequality on the above chain is valid for sufficiently large j. The reason is that when j is sufficiently large, $f_j((D_1)_k) = (D_2)^j_k \subset \widetilde{D}$ by Theorem (e), where $\widetilde{D} = V \cap D_2$, $p \in V$ is an open set in $\mathbf{C}^n$. It follows that for fixed k and large j, we have the chain

$$\frac{C^n_{(D_1)_k}(x,v)}{E^n_{D_1}(x,v)} \geq \frac{C^n_{(D_2)^j_k}(x_j, df_j(v))}{E^n_{D_2}(x_j, df_j(v))} \geq \frac{C^n_{\widetilde{D}}(x_j, df_j(v))}{E^n_{D_2}(x_j, df_j(v))}$$

of inequalities (Theorem (a)(5) has been used here).

Observe that:

(i) By the volume decreasing property under holomorphic mappings (Theorem (a)(2)), we have $E^n_{\widetilde{D}}(x_j, \mathrm{d}f_j(v)) \geq E^n_{D_2}(x_j, \mathrm{d}f_j(v))$ as the inclusion map $\widetilde{D} \hookrightarrow D_2$ is holomorphic. Therefore, we have

$$\frac{C^n_{(D_1)_k}(x,v)}{E^n_{D_1}(x,v)} \geq \frac{C^n_{\widetilde{D}}(x_j, \mathrm{d}f_j(v))}{E^n_{\widetilde{D}}(x_j, \mathrm{d}f_j(v))}.$$

(ii) Again by the strong pseudoconvexity of $p \in \partial D_2$, one obtains

$$\frac{C^n_{\widetilde{D}}(x_j, \mathrm{d}f_j(v))}{E^n_{\widetilde{D}}(x_j, \mathrm{d}f_j(v))} \to 1 \quad \text{as } x_j \to p$$

by Theorem (b).

(iii) If we let $k \to \infty$, then $C^n_{(D_1)_k}(x,v) \to C^n_{(D_1)}(x,v)$. This approximation property can be proved by elementary normal family argument.

(iv) It is always true that $C^n_{(D_1)}(x,v)/E^n_{(D_1)}(x,v) \leq 1$ by Theorem (a)(1).

Combining (i)–(iv), and letting $j \to \infty$ and then $k \to \infty$, one concludes that $1 \geq C^n_{(D_1)}(x,v)/E^n_{(D_1)}(x,v) \geq 1$, proving our claim.

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