

TOPOLOGICAL TRIVIALITY OF A FAMILY OF ZERO-SETS

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ABSTRACT. This paper gives conditions on a mapping $F: U \times \mathbf{R}^k \rightarrow \mathbf{K}^p$ ($U \subset \mathbf{K}^n$ open, $\mathbf{K} = \mathbf{R}$ or \mathbf{C}) such that the family $F_t^{-1}(0)$ is a topologically trivial family (i.e., does not change topologically as $t \in \mathbf{R}^k$ varies). As an application an easy proof is given of a counterexample to a conjecture of Thom concerning the number of topologically different realizations of a given jet.

1. Introduction. Let \mathbf{K} denote the field of real or complex numbers and let U be a neighborhood of 0 in \mathbf{K}^n . Given a continuous map $F: U \times \mathbf{R}^k \rightarrow \mathbf{K}^p$ and t in \mathbf{R}^k , we define $Z(F) = F^{-1}(0)$, $F_t: U \rightarrow \mathbf{K}^p$, $F_t = F(\cdot, t)$ and $Z(F)_t = F_t^{-1}(0)$. We think of t, F and $Z(F)$ as parameter, family of maps and family of varieties, respectively. The goal of this note is to give sufficient conditions for $Z(F)$ (respectively F) to be locally topologically trivial at the origin. More precisely, given a point t_0 in \mathbf{R}^k and a relatively compact neighborhood V_0 of t_0 , we are interested in finding a neighborhood U_0 of 0 in U and a continuous map $\sigma: U_0 \times V_0 \rightarrow U$ such that for each t in V_0 the map $\sigma_t = \sigma(\cdot, t)$ sends 0 onto 0, homeomorphically transforms U_0 onto $\sigma_t(U_0)$, $\sigma_t^{-1}(Z(F)_t) = U_0 \cap Z(F)_{t_0}$ (respectively $F_t \circ \sigma_t = F_{t_0}$ on U_0) and σ_{t_0} is the identity map from U_0 onto itself.

Even in the best understood case, for $p = 1$ and under strong conditions imposed on F , the problem is far from solved. Assume that F is of class C^∞ and for each t in \mathbf{R}^k , $F_t: U \rightarrow \mathbf{K}$ is a polynomial function having an isolated critical point at 0. Then, by the results of Lê and Ramanujam [8] and Timourian [12], respectively, the families $Z(F)$ and F are locally topologically trivial at the origin, provided $\mathbf{K} = \mathbf{C}$ and $n \neq 3$. The corresponding result holds in the real case, thanks to King's theorem [5], for $n \leq 3$ and is false for all $n \geq 5$. Clearly, in the real case, it is necessary to impose further restrictions on F . Following other authors, Damon [3, 4], Oka [9], we consider here families which are deformations of a quasi-homogeneous polynomial map (cf. §2 for details).

Other papers related to this one are those of Buchner et al. [2] and Percell and Brown [10].

The main results are Theorems 2.3 and 2.5. §2 also contains some examples.

2. Deformations of quasi-homogeneous maps. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n -tuple of positive integers and let γ be a positive integer. A polynomial function $P: \mathbf{K}^n \rightarrow \mathbf{K}$ is said to be of type $(\alpha; \gamma)$ if P is a \mathbf{K} -linear combination of monomials $x_1^{i_1} \cdots x_n^{i_n}$ for which $\alpha_1 i_1 + \cdots + \alpha_n i_n = \gamma$. Given an analytic function-germ $f: (\mathbf{K}^n, 0) \rightarrow (\mathbf{K}, 0)$, we define $w(f, \alpha)$ to be the minimum of the

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integers $\alpha_1 i_1 + \dots + \alpha_n i_n$ such that the monomial $x_1^{i_1} \dots x_n^{i_n}$ appears in the Taylor expansion of f at 0 with a nonzero coefficient. Now let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_p)$ be an n -tuple and a p -tuple of positive integers. A polynomial map $Q = (Q_1, \dots, Q_p): \mathbf{K}^n \rightarrow \mathbf{K}^p$ is said to be of type $(\alpha; \beta)$ if for each $j = 1, \dots, p$, Q_j is of type $(\alpha; \beta_j)$.

We need two more concepts.

DEFINITION 2.1. A polynomial map $Q: \mathbf{K}^n \rightarrow \mathbf{K}^p$ is said to be nondegenerate (resp. strongly nondegenerate) if for all x in $Q^{-1}(0) \setminus \{0\}$ (resp. $\mathbf{K}^n \setminus \{0\}$) the derivative $d_x Q$ is surjective.

Note that if $Q^{-1}(0) \setminus \{0\} \neq \emptyset$, then nondegeneracy requires $n \geq p - 1$.

DEFINITION 2.2. A map $F: U \times \mathbf{R}^k \rightarrow \mathbf{K}^p$, where U is a neighborhood of 0 in \mathbf{K}^n , is called admissible (resp. strongly admissible) for $(\alpha; \beta)$ if F can be written as $F = Q + G$ with $Q, G: U \times \mathbf{R}^k \rightarrow \mathbf{K}^p$ satisfying the following conditions:

(i) the j th components Q_j and G_j of Q and G , resp., are of class C^2 , G_j , $\partial G_j / \partial t_s, \partial^2 G_j / \partial t_s \partial t_q, s, q = 1, \dots, k$, are of class C^{r_j} in the x -variable, where $r_j > \max\{\beta_j / \alpha_1, \dots, \beta_j / \alpha_n\}$ and all derivatives in the x variable up to order r_j of $G_j, \partial G_j / \partial t_s, \partial^2 G_j / \partial t_s \partial t_q$ are C^0 in (x, t) ;

(ii) for each t in $\mathbf{R}^k, Q_t = Q(\cdot, t)$ is the restriction to U of a nondegenerate (resp. strongly nondegenerate) map of type $(\alpha; \beta)$ and, moreover, $G_t = G(\cdot, t)$ is holomorphic if $\mathbf{K} = \mathbf{C}$;

(iii) $w(T^{r_j-1} G_j(\cdot, t), \alpha) > \beta_j$ for all t in \mathbf{R}^j and $j = 1, \dots, p$, where $T^r(\cdot)$ denotes the Taylor polynomial (with respect to the variable in \mathbf{K}^n) at 0 of degree r . (Note that $w(T^{r_j} G_j(\cdot, t), \alpha) > \beta_j$ is also satisfied, since, by the inequality in (i), $w(x_1^{i_1} \dots x_n^{i_n}) > \beta_j$ if $i_1 + \dots + i_n \geq r_j$.)

Our main result is the following.

THEOREM 2.3. *Let $F: U \times \mathbf{R}^k \rightarrow \mathbf{K}^p$ be an admissible (resp. strongly admissible) map for $(\alpha; \beta)$. Then given a point t_0 in \mathbf{R}^k and a relatively compact neighborhood V_0 of t_0 in \mathbf{R}^k , there exists a neighborhood U_0 of 0 in \mathbf{K}^n and a continuous map $\sigma: U_0 \times V_0 \rightarrow U$ such that for each t in V_0, σ_t sends 0 onto 0, transforms homeomorphically U_0 onto $\sigma_t(U_0), \sigma_t^{-1}(F_t^{-1}(0)) = U_0 \cap F_{t_0}^{-1}(0)$ (resp. $F_t \circ \sigma_t = F_{t_0}$ on U_0) and σ_{t_0} is the identity map from U_0 onto itself. In other words, the family of varieties $Z(F)$ (resp. the family of maps F) is locally topologically trivial at the origin.*

Now we shall point out one application of Theorem 2.3. It is known that Thom's conjecture [11], stating that an r -jet w in $J^r(n, p)$ admitting two topologically different realizations (i.e., for which one can find two map-germs $f, g: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ whose r -jets are equal to w and g is always different from $\tau \circ f \circ \sigma$, where $\sigma: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and $\tau: (\mathbf{R}^p, 0) \rightarrow (\mathbf{R}^p, 0)$ are local homeomorphisms) has actually infinitely many topologically different realizations, is true if $p = 1$ and C^r or C^{r+1} realizations are allowed [1]. The conjecture is false if we only allow C^{r+2} realizations [7]. In the example that follows we show that the arguments of [7], involving the complexifications, computation of Milnor numbers and a delicate theorem of King [5], can be simplified by applying Theorem 2.3.

EXAMPLE 2.4. Consider the 6-jet $w = x_1^3 - 3x_1x_2^5$ (we identify an r -jet with its unique polynomial realization of degree not exceeding r). Let $f: (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}, 0)$ be a C^8 function-germ whose 6-jet is equal to w . We claim that there exists a local

homeomorphism $\sigma: (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ such that $f \circ \sigma = w + cx_2^7$ for some c in \mathbf{R} and a local homeomorphism $\sigma_c: (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ for which $(w + cx_2^7) \circ \sigma_c = x_1^3 + x_2^7$ if $c \neq 0$. Note that since the set-germs at 0 of $\{x \in \mathbf{R}^2 | w(x) = 0\}$ and $\{x \in \mathbf{R}^2 | x_1^3 + x_2^7 = 0\}$ are not homeomorphic, w admits exactly two topologically different C^8 realizations. To prove the claim define $F(x, t) = w(x) + cx_2^7 + t(f(x) - w(x) - cx_2^7)$, where c is the coefficient of x_2^7 in the Taylor expansion of f of order 7. Note that F is strongly admissible for $((5, 2; 15)$ if $c = 0$ (with $Q(x, t) = w(x)$ and $G(x, t) = t(f(x) - w(x))$) and for $((7, 3; 21)$ if $c \neq 0$ (with $Q(x, t) = x_1^3 + cx_2^7$ and $G(x, t) = -3x_1x_2^5 + t(f(x) - w(x) - cx_2^7)$). In both cases there exists a local homeomorphism $\sigma: (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ with $f \circ \sigma = F_1 \circ \sigma = F_0 = w + cx_2^7$. If $c \neq 0$, define $H(x, t) = x_1^3 + cx_2^7 - 3tx_1x_2^5$. Since H is strongly admissible for $((7, 3; 21)$ (with $Q(x, t) = x_1^3 + cx_2^7$ and $G(x, t) = -3tx_1x_2^5$), there exists a local homeomorphism $\tau_c: (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ satisfying $(w + cx_2^7) \circ \tau_c = H_1 \circ \tau_c = H_0 = x_1^3 + cx_2^7$. It suffices to take $\sigma_c = \tau_c \circ \gamma$, where $\gamma(x) = (x_1, c^{-1/7}x_2)$.

Let q and m be nonnegative integers with $q + m = n$. Denote by y and λ variables in \mathbf{K}^q and \mathbf{K}^m , respectively.

THEOREM 2.5. *If condition (ii) in Definition 2.2 is replaced by*

(ii') *for each t in \mathbf{R}^k , Q_t is the restriction to U of a polynomial map of type $(\alpha; \beta)$, the partial derivative of Q_t with respect to y is surjective for all (y, λ) in $Q_t^{-1}(0) \setminus \{0\}$ (resp. $\mathbf{K}^n \setminus \{0\}$) and, moreover, G_t is holomorphic if $\mathbf{K} = \mathbf{C}$, then under the assumptions of Theorem 2.3 each homeomorphism σ_i is of the form $\sigma_i(y, \lambda) = (\bar{\sigma}_i(y, \lambda), \lambda)$.*

After giving the proof of 2.3 we will explain the modifications needed to prove 2.5.

The following example shows that a result in [10] can be improved upon by application of Theorem 2.5.

EXAMPLE 2.6 [10]. Let $f(y_1, y_2, \lambda) = (y_2 + \lambda y_1, y_1^2 + \lambda y_2)$ and $g = f + h$, where $h = (h_1, h_2)$, h_1 is of class C^4 and h_2 is of class C^5 . Assume that $w(T^3h_1, (2, 3, 1)) > 3$ and $w(T^4h_2, (2, 3, 1)) > 4$. Then there exists a local homeomorphism $\sigma: (\mathbf{R}^2 \times \mathbf{R}, 0) \rightarrow (\mathbf{R}^2 \times \mathbf{R}, 0)$ of the form $\sigma(y, \lambda) = (\bar{\sigma}(y, \lambda), \lambda)$ such that $f^{-1}(0) = \sigma(g^{-1}(0))$. To show this it suffices to apply Theorem 2.5 to the family $F(y, \lambda, t) = f(y, \lambda) + th(y, \lambda)$ which is admissible for $((2, 3, 1); (3, 4))$ (with (ii') in Definition 2.2 instead of (ii)). Note that [10] gives the same conclusion under the stronger assumptions $T^3h_1 = 0$ and $T^4h_2 = 0$.

3. The proof of Theorem 2.3. This section is devoted to proving Theorem 2.3. Clearly, the complex case follows from the real one, so we assume $\mathbf{K} = \mathbf{R}$. Without loss of generality we may assume that $t_0 = 0$ and that $V_0 =$ the cube $(-L, L)^k$ for some positive real number L . Let S^{n-1} be the unit sphere in \mathbf{R}^n and let $\varepsilon > 0$ be chosen in such a way that the map $\phi: S^{n-1} \times (-\varepsilon, \varepsilon) \rightarrow \mathbf{R}^n$ defined by

$$\psi(x_1, \dots, x_n, r) = (r^{\alpha_1}x_1, \dots, r^{\alpha_n}x_n)$$

has values in U .

For each $j = 1, \dots, p$ define $H_j: S^{n-1} \times (-\varepsilon, \varepsilon) \times \mathbf{R}^k \rightarrow \mathbf{R}$ by $H_j(x, r, t) = r^{-\beta_j}G_j(\phi(x, r), t)$ for $r \neq 0$ and $H_j(x, 0, t) = 0$. Conditions (i) and (iii) of Definition 2.2 imply that $H_j, \partial H_j / \partial x_l$ and $\partial H_j / \partial t_s, l = 1, \dots, n, s = 1, \dots, k$, are at least of class C^1 and are equal to 0 if $r = 0$ (this is obvious if G_j is of class C^∞ , for

then H_j , $\partial H_j/\partial x_l$ and $\partial H_j/\partial t_s$ are also of class C^∞ and it requires a lengthy but straightforward computation if G_j merely satisfies the differentiability conditions (i). Let E_{ij} be the $p \times p$ matrix whose entry in the i th row and j th column is equal to 1 and all other entries are equal to 0. Let $\pi: S^{n-1} \times (-\varepsilon, \varepsilon) \times \mathbf{R}^k \rightarrow S^{n-1} \times \mathbf{R}^k$ be the projection and let A be the $p \times (p^2 + n)$ matrix whose columns consist of the columns $E_{ij} \cdot (Q \circ \pi + H)$ (product of $p \times p$ matrix by $p \times 1$ matrix) and the columns of the $p \times n$ matrix $C = ((\partial Q_j/\partial x_l) \circ \pi + \partial H_j/\partial x_l)$. Finally we denote by A^* the matrix whose columns consist of the columns $E_{ij} \cdot (Q \circ \pi)$ and the columns of $C^* = ((\partial Q_j/\partial x_l) \circ \pi)$. (In the above, $i, j = 1, \dots, p$ and $l = 1, \dots, n$.)

Note that all entries of A are C^1 functions in the variables (x, r, t) . By the nondegeneracy condition, A^* has rank p at each point in $S^{n-1} \times (-\varepsilon, \varepsilon) \times \mathbf{R}^k$ and hence A has the same property when ε is chosen suitably small and \mathbf{R}^k is replaced by a compact set which we take to be $[-L - \delta, L + \delta]^k$. It follows that the column $((\partial Q/\partial t_k) \circ \pi + (\partial H/\partial t_k))$ is a linear combination of the columns of the matrix A with coefficients which are C^1 functions on $S^{n-1} \times (-\varepsilon, \varepsilon) \times (-L - \delta, L + \delta)^k$. Thus

$$(3.1) \quad \frac{\partial Q}{\partial t_k} \circ \pi + \frac{\partial H}{\partial t_k} = B \cdot (Q \circ \pi + H) + C \cdot u$$

where B is a $p \times p$ matrix with C^1 entries and u is an n -vector with C^1 components u_1, \dots, u_n . Note that if F is strongly admissible, then, by the same argument as above, the matrix C has rank p at each point in $S^{n-1} \times (-\varepsilon, \varepsilon) \times [-L - \delta, L + \delta]$ and we may assume that B is the zero matrix. Now multiply the i th component of (3.1) by r^{β_i} . By definition of H_j ,

$$(3.2) \quad \frac{\partial F}{\partial t_k}(\psi(x, r), t) = \tilde{B}(x, r, t) \cdot F(\psi(x, r), t) + d_{\psi(x, r)} F_t(w(x, r), t)$$

on $S^{n-1} \times ((-\varepsilon, 0) \cup (0, \varepsilon)) \times (-L - \delta, L + \delta)^k$ where $\tilde{B}(x, r, t)$ is a $p \times p$ matrix with entries which are C^1 on $S^{n-1} \times ((-\varepsilon, 0) \cup (0, \varepsilon)) \times (-L - \delta, L + \delta)^k$ and where the i th component $w_i(x, r, t)$ of $w(x, r, t)$ is equal to $r^{\alpha_i} u_i(x, r, t)$ which is C^1 on $S^{n-1} \times (-\varepsilon, \varepsilon) \times (-L - \delta, L + \delta)^k$.

Now (3.2) can be rewritten as

$$(3.3) \quad \frac{\partial F}{\partial t_k}(x, t) = \tilde{B}(\psi^{-1}(x), t) F(x, t) + d_x F_t(w(\psi^{-1}(x), t))$$

where $x \in \psi(S^{n-1} \times (-\varepsilon, \varepsilon)) \setminus \{0\}$. We can regard $w(\psi^{-1}(x), t)$ as a vector field on $\psi(S^{n-1} \times (-\varepsilon, \varepsilon)) \setminus \{0\}$. We choose to regard it as a time dependent vector field; with t_k being the time variable and (t_1, \dots, t_{k-1}) being a parameter. We wish to integrate this vector field, i.e. solve

$$(3.4) \quad \begin{cases} \partial \tau^k(x, t)/\partial t_k = -w_t \circ \psi^{-1}(\tau^k(x, t)), \\ \tau^k(x, t_1, \dots, t_{k-1}, 0) = x. \end{cases}$$

Suppose, for the moment, that ε_k, δ_k and τ_k can be found such that $0 < \varepsilon_k < \varepsilon$, $0 < \delta_k < \delta$ and τ^k is a mapping from

$$(\psi(S^{n-1} \times (-\varepsilon_k, \varepsilon_k)) \setminus \{0\}) \times (-L - \delta_k, L + \delta_k)^k \rightarrow \psi(S^{n-1} \times (-\varepsilon, \varepsilon)) \setminus \{0\}$$

satisfying (3.4). Let

$$T^k: (\psi(S^{n-1} \times (-\varepsilon_k, \varepsilon_k)) \setminus \{0\}) \times (-L - \delta_k, L + \delta_k)^k \rightarrow GL(p, \mathbf{R})$$

be the solution to

$$(3.5) \quad \begin{cases} \partial T^k(x, t) / \partial t_k = -T^k(x, t) \cdot \tilde{B}(\psi^{-1}(\tau^k(x, t)), t), \\ T^k(x, t_1, \dots, t_{k-1}, 0) = E(\text{identity } p \times p \text{ matrix}). \end{cases}$$

The equations (3.3), (3.4) and (3.5) together imply

$$\frac{\partial}{\partial t_k} (T^k(x, t) \cdot F(\tau^k(x, t), t)) = 0$$

from which it follows that

$$T^k(x, t) \cdot F(\tau^k(x, t), t) = F(x, t_1, \dots, t_{k-1}, 0).$$

In case F is strongly admissible, $T^k(x, t)$ can be taken to be E because in that case B , and hence \tilde{B} , can be taken to be the zero matrix.

It will also be shown, in the course of the construction of $\tau^k(x, t)$, that if $\tau^k(0, t)$ is defined to be 0 then $\tau_t^k: \psi(S^{n-1} \times (-\varepsilon_k, \varepsilon_k)) \rightarrow \psi(S^{n-1} \times (-\varepsilon, \varepsilon))$ is a homeomorphism.

Granted this, (2.3) follows by induction with respect to the number of parameters.

To prove the claims made about τ^k it is convenient to lift the vector field $w_t \circ \psi^{-1}$ to $S^{n-1} \times ((-\varepsilon, 0) \cup (0, \varepsilon))$. It will be shown that the lifted vector field has a C^1 extension to $S^{n-1} \times (-\varepsilon, \varepsilon)$ which is tangent to $S^{n-1} \times \{0\}$ at all points of $S^{n-1} \times \{0\}$. Assume this has been done. Denote the lifted vector field by ξ_t . Consider solving the equations

$$(3.6) \quad \begin{cases} \partial \tilde{\tau}^k(x, r, t) / \partial t_k = -\xi_t(\tilde{\tau}^k(x, r, t)), \\ \tilde{\tau}^k(x, r, t_1, \dots, t_{k-1}, 0) = (x, r). \end{cases}$$

This has a C^1 solution on $S^{n-1} \times (-\varepsilon_k, \varepsilon_k) \times (-L - \delta_k, L + \delta_k)^k$ for suitable $0 < \varepsilon_k < \varepsilon$ and $0 < \delta_k < \delta$. Furthermore, $\tilde{\tau}_t^k$ is a C^1 embedding $S^{n-1} \times (-\varepsilon_k, \varepsilon_k) \rightarrow S^{n-1} \times (-\varepsilon, \varepsilon)$ such that

$$\tilde{\tau}_t^k(S^{n-1} \times \{0\}) = S^{n-1} \times \{0\}, \quad \tilde{\tau}_t^k(S^{n-1} \times (0, \varepsilon_k)) \subset S^{n-1} \times (0, \varepsilon)$$

and

$$\tilde{\tau}_t^k(S^{n-1} \times (-\varepsilon_k, 0)) \subset S^{n-1} \times (-\varepsilon, 0).$$

If we define τ_t^k by $\tau_t^k = \psi \circ \tilde{\tau}_t^k \circ \psi^{-1}$ on $\psi(S^{n-1} \times (0, \varepsilon_k))$ then τ_t^k satisfies (3.4) and has all the desired properties.

Only the C^1 extension of the lifting remains to be shown. Recall that

$$w_t(x, r) = (r^{\alpha_1} u_1(x, r, t), \dots, r^{\alpha_n} u_n(x, r, t))$$

where each u_i is C^1 on $S^{n-1} \times (-\varepsilon, \varepsilon) \times (-L - \delta, L + \delta)$. The vector field ξ_t is the solution of $d_{(x,r)}\psi(\xi_t) = w_t(x, r)$. If we identify the tangent bundle of S^{n-1} with a subbundle of the trivial bundle $S^{n-1} \times \mathbf{R}^n$ then the equation for ξ_t becomes

$$\begin{bmatrix} r^{\alpha_1} & 0 & \dots & 0 & \alpha_1 x_1 r^{\alpha_1-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & r^{\alpha_n} & \alpha_n x_n r^{\alpha_n-1} \\ x_1 & x_2 & \dots & x_n & 0 \end{bmatrix} \begin{bmatrix} \xi_{1t} \\ \vdots \\ \xi_{nt} \\ \xi_{(n+1)t} \end{bmatrix} = \begin{bmatrix} r^{\alpha_1} u_1(x, r, t) \\ \vdots \\ r^{\alpha_n} u_n(x, r, t) \\ 0 \end{bmatrix}.$$

In solving for each ξ_{it} by Cramer's rule we see that the denominator is the product of $r^{(\sum_i \alpha_i)-1}$ and a nonvanishing term and the numerator has a factor of $r^{(\sum_i \alpha_i)-1}$.

This proves that ξ_t has the claimed C^1 extension. Finally we observe that in using Cramer's rule for ξ_{n+1} we obtain a factor of $r^{\sum_i \alpha_i}$ in the numerator, thus justifying the claim that the extension is tangent to $S^{n-1} \times \{0\}$.

4. The proof of Theorem 2.5. We proceed exactly as in the proof of 2.3 except that we make the following additional observtion: The $p \times (p^2 + q)$ matrix whose columns consist of the columns $E_{ij} \cdot (Q \circ \pi)$ and the columns of $((\partial Q_j / \partial y_l) \circ \pi)$ ($i, j = 1, \dots, p$ and $l = 1, \dots, q$) has rank p . Hence we will again obtain (3.1) but can assume u_{q+1}, \dots, u_n are all identically zero. When we solve (3.4) (with $x = (y, \lambda)$), we will find the last m components of r^k are constants and must therefore be λ , by the assumption of the initial condition.

5. Alternative proof of 2.3 suggested by referee. We now outline a second approach to Theorem 2.3 suggested by the referee and using the papers [5 and 6] by H. King. We consider $F(\phi(x, r), t) = F(r^{\alpha_1} x_1, \dots, r^{\alpha_n} x_n, t)$ with $x = (x_1, \dots, x_n) \in S^{n-1}$ and $r \geq 0$. By (iii) we can write

$$F(r^{\alpha_1} x_1, \dots, r^{\alpha_n} x_n, t) = (r^{\beta_1} Q_1(x, t) + r^{\beta_1+1} \tilde{G}_1(x, r, t), \dots, r^{\beta_n} Q_n(x, t) + r^{\beta_n+1} \tilde{G}_n(x, r, t))$$

where $\tilde{G}_j(x, r, t)$ is C^{r_j} on $S^{n-1} \times (0, \infty)$ for each t and C^0 on $S^{n-1} \times [0, \infty) \times \mathbf{R}^k$. It follows that if $t \in V_0$, $F(\phi(x, r), t) = 0$ and r is small then x is close on S^{n-1} to a zero of Q_t . Now, from (ii) we have $d_{(x,r)}(Q_t \circ \phi): T_{(x,r)}(S^{n-1} \times (0, \infty)) \rightarrow \mathbf{R}^p$ is surjective if $(Q_t \circ \phi)(x, r) = 0$. From this and the quasihomogeneity of Q_t , it follows that $d_x Q_t: T_x S^{n-1} \rightarrow \mathbf{R}^p$ is surjective whenever $Q_t(x) = 0$. Hence, for small r , we have $d_x Q_t$ is surjective whenever $t \in V_0$ and $F(\phi(x, r), t) = 0$. Next, we compute

$$d_x(F_t \circ \phi_r) = (r^{\beta_1} d_x Q_{1,t} + r^{\beta_1+1} d_x \tilde{G}_{1,r,t}, \dots, r^{\beta_n} d_x Q_{n,t} + r^{\beta_n+1} d_x \tilde{G}_{n,r,t}).$$

Now it follows from the assumptions that $d_x \tilde{G}_{j,r,t}$ extends continuously to $r = 0$ and hence is bounded for r small, $t \in V_0$ and $x \in S^{n-1}$. We conclude that $d_x(F_t \circ \phi_r)$ is surjective for r small, $t \in V_0$ and $x \in S^{n-1}$ and $F_t \circ \phi_r(x) = 0$. This shows that $\{(x, r) \in S^{n-1} \times (0, \infty) | F_t \circ \phi_r(x) = 0\}$ is a submanifold of $S^{n-1} \times (0, \infty)$ when r is small and $t \in V_0$ and this submanifold is transverse to each $S^{n-1} \times \{r\}$.

In \mathbf{R}^n the above argument shows the following for the mapping F : There is a neighborhood of 0 in \mathbf{R}^n of the form $U = \{(x_1, \dots, x_n) | (x_1/\varepsilon^{\alpha_1})^2 + \dots + (x_n/\varepsilon^{\alpha_n})^2 < 1\}$ such that for every $t \in V_0$, $(F_t^{-1}(0) \cap U) \setminus \{0\}$ is a submanifold of $\mathbf{R}^n \setminus \{0\}$ transversal to every ellipsoid of the form $\{(x_1, \dots, x_n) | (x_1/r^{\alpha_1})^2 + \dots + (x_n/r^{\alpha_n})^2 = 1\}$ for $r < \varepsilon$ and such that $(F^{-1}(0) \cap (U \times V_0)) \setminus \{0\} \times V_0$ is a C^2 submanifold.

Let now $r(x_1, \dots, x_n)$ be the r -component of the inverse of $\phi(x_1, \dots, x_n, r) = (r^{\alpha_1} x_1, \dots, r^{\alpha_n} x_n)$. The transversality assertion is equivalent to the assertion that r restricted to $(F_t^{-1}(0) \cap U) \setminus \{0\}$ has no critical points. Consequently, the map $(U \setminus \{0\}) \times V_0 \rightarrow (0, \varepsilon) \times V_0$ given by $(x, t) \rightarrow (r(x), t)$ is a C^2 submersion which restricts to a C^2 submersion from $(F^{-1}(0) \cap (U \times V_0)) \setminus \{0\} \times V_0$ to $(0, \varepsilon) \times V_0$. Since the submersion is also proper it follows from the Ehresmann fibration theorem that $(U \setminus \{0\}) \times V_0$ is a locally trivial fibration over $(0, \varepsilon) \times V_0$ in such a way that $(F^{-1}(0) \cap (U \times V_0)) \setminus \{0\} \times V_0$ is a sublocally-trivial fibration over $(0, \varepsilon) \times V_0$. There is no loss in generality in assuming V_0 is contractible. Hence, these fibrations are actually topologically trivial. Choose $0 < r_0 < \varepsilon$ and let E_{r_0} be the ellipsoid

$\{(x_1, \dots, x_n) \mid \sum_i (x_i/r_0^{\alpha_i})^2 = 1\}$. There is a homeomorphism $\phi: (U \setminus \{0\}) \times V_0 \rightarrow (0, \varepsilon) \times E_{r_0} \times V_0$ of the form $(x, t) \rightarrow (r(x), \xi(x, t), t)$ which restricts to a homeomorphism

$$F^{-1}(0) \cap (U \times V_0) \setminus \{0\} \times V_0 \rightarrow (0, \varepsilon) \times (F_{t_0}^{-1}(0) \cap E_{r_0}) \times V_0.$$

So, the mapping $\gamma_t: x \mapsto (r(x), \xi_t(x))$ is a homeomorphism $U \setminus \{0\} \rightarrow (0, \varepsilon) \times E_{r_0}$ mapping $(F_t^{-1}(0) \cap U) \setminus \{0\}$ onto $(0, \varepsilon) \times F_{t_0}^{-1}(0) \cap E_{r_0}$. It suffices now to define σ_t by $\gamma_t^{-1} \circ \gamma_{t_0}$ on $U \setminus \{0\}$ and $\sigma_t(0) = 0$.

The above argument proves the admissible version of 2.3. The strongly admissible version now follows by applying Theorem 1 of the paper [5] by H. King, which gives a criterion (easily seen to be satisfied in our case) for when topological triviality of the family of germs of zero sets is equivalent to topological triviality of the family of germs of the mappings.

It is not clear to us if it is possible to modify the above procedure to give a proof of 2.5.

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