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TWO MIXED HADAMARD TYPE GENERALIZATIONS OF HEINZ INEQUALITY

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Dedicated to Professor Zirô Takeda on his 65th birthday with respect and affection

ABSTRACT. We give two types of mixed Hadamard inequalities containing the terms T, $|T|$, and $|T^*|$, where T is a bounded linear operator on a complex Hilbert space. As an immediate consequence of these results, we can easily show some extensions of the Hadamard inequality and also the Heinze inequality:

$$(*) \qquad |(Tx,y)|^2 \leq (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)$$

for any T, any x,y in H, and any real number α with $0 \leq \alpha \leq 1$. And the following conditions are equivalent in case $0 < \alpha < 1$:

(1) the equality in $(*)$ holds;
(2) $|T|^{2\alpha}x$ and T^*y are linearly dependent;
(3) Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent.

Results in this paper would remain valid for unbounded operators under slight modifications.

1. Introduction. An operator T means a bounded linear operator on a complex Hilbert space H. We give two types of mixed Hadamard theorems containing the terms T, $|T|$, and $|T^*|$. These results are some extensions of Hadamard's theorem and the Heinz inequality

$$(\mathrm{I}_3) \qquad |(Tx,y)|^2 \leq (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)$$

for any T, any x,y in H, and any real number α with $0 \leq \alpha \leq 1$. Also we scrutinize the cases when the equalities in these mixed Hadamard theorems hold. As an immediate consequence of this scrutiny, we can show that the equality in (I_3) holds if and only if $|T|^{2\alpha}x$ and T^*y are linearly dependent if and only if Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent in case α with $0 < \alpha < 1$. Since (I_3) is equivalent to (I_4)

$$(\mathrm{I}_4) \qquad |(Tx,y)| \leq \||T|^{\alpha}x\| \, \||T^*|^{1-\alpha}y\|,$$

so that one might believe that the equality in (I_3) or (I_4) would hold if and only if $|T|^{2\alpha}x$ and $|T^*|^{2(1-\alpha)}y$ are linearly dependent or $|T|^{\alpha}x$ and $|T^*|^{1-\alpha}y$ are linearly dependent. But here we can give a simple counter example to these mistakes. By this reason, the form of (I_3) is more convenient than (I_4) in order to remind us of the case when the equality in (I_3) or (I_4) holds. This paper extends the results obtained in [**3** and **4**].

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2. Statement of the results.

THEOREM 1 (MIXED HADAMARD TYPE 1). *For any operator T on H, any $x_1, x_2, \ldots, x_n$ in H, and any nonnegative number α and β with $\alpha+\beta=1$, let G_n be defined by*

$$G_n = \begin{vmatrix} (|T|^{2\alpha}x_1, x_1) & (Tx_1, x_2) & (Tx_1, x_3) & \cdots & (Tx_1, x_n) \\ (T^*x_2, x_1) & (|T^*|^{2\beta}x_2, x_2) & (|T^*|^{2\beta}x_2, x_3) & \cdots & (|T^*|^{2\beta}x_2, x_n) \\ (T^*x_3, x_1) & (|T^*|^{2\beta}x_3, x_2) & (|T^*|^{2\beta}x_3, x_3) & \cdots & (|T^*|^{2\beta}x_3, x_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (T^*x_n, x_1) & (|T^*|^{2\beta}x_n, x_2) & (|T^*|^{2\beta}x_n, x_3) & \cdots & (|T^*|^{2\beta}x_n, x_n) \end{vmatrix}.$$

Then

$$(\mathrm{I}_1) \qquad 0 \le G_n \le \||T|^{\alpha}x_1\|^2 \prod_{j=2}^{n} \||T^*|^{\beta}x_j\|^2.$$

Let $S_1(\alpha)$ and $S_2(\beta)$ be two systems of vectors defined by

$$S_1(\alpha) = \{|T|^{2\alpha}x_1, T^*x_2, T^*x_3, \ldots, T^*x_n\}$$

and

$$S_2(\beta) = \{Tx_1, |T^*|^{2\beta}x_2, |T^*|^{2\beta}x_3, \ldots, |T^*|^{2\beta}x_n\}.$$

(1) $0<\alpha<1$. (i) $G_n = 0$ *iff $S_1(\alpha)$ is a system of linearly dependent vectors iff $S_2(\beta)$ is a system of linearly dependent vectors.*

(ii) *The following conditions* (a) *and* (b) *are equivalent:*
- (a) *the right-hand side equality in* (I_1) *holds,*
- (b) *the following* (b_1) *or* (b_2) *holds:*
- (b_1) $(Tx_1, x_j) = 0$ *for* $j = 2, 3, \ldots, n$ *and* $(|T^*|^{2\beta}x_j, x_k) = 0$ *for* $j<k$ $(j = 2, 3, \ldots, n-1)$.
- (b_2) $S_1(\alpha)$ *contains the zero vector* (*equivalently,* $S_2(\beta)$ *contains the zero vector*).

(2) $\alpha = 1$. (i) $G_n = 0$ *iff* $S_2(0) = \{Tx_1, x_2, x_3, \ldots, x_n\}$ *is a system of linearly dependent vectors.*

(ii) *The following conditions* (a) *and* (b) *are equivalent:*
- (a) *the right-hand side equality in* (I_1) *holds,*
- (b) *the following* (b_1) *or* (b_2) *holds:*
- (b_1) $(Tx_1, x_j) = 0$ *for* $j = 2, 3, \ldots, n$ *and* $(x_j, x_k) = 0$ *for* $j<k$ $(j = 2, 3, \ldots, n-1)$.
- (b_2) $S_2(0)$ *contains the zero vector.*

(3) $\alpha = 0$. (i) $G_n = 0$ *iff* $S_1(0) = \{x_1, T^*x_2, T^*x_3, \ldots, T^*x_n\}$ *is a system of linearly dependent vectors.*

(ii) *The following conditions* (a) *and* (b) *are equivalent:*
- (a) *the right-hand side equality in* (I_1) *holds,*
- (b) *the following* (b_1) *or* (b_2) *holds:*
- (b_1) $(x_1, T^*x_j) = 0$ *for* $j = 2, 3, \ldots, n$ *and* $(|T^*|^2x_j, x_k) = 0$ *for* $j<k$ $(j = 2, 3, \ldots, n-1)$.
- (b_2) $S_1(0)$ *contains the zero vector.*

THEOREM 2 (MIXED HADAMARD TYPE 2). *For any operator T on H, any $x_1, x_2, \ldots, x_n$ in H, and any nonnegative number α and β with $\alpha+\beta=1$, let G_{2n} be defined by*

$$G_{2n}=\begin{vmatrix} (|T|^{2\alpha}x_1,x_1) & (Tx_1,x_2) & \ldots & (|T|^{2\alpha}x_1,x_{2n-1}) & (Tx_1,x_{2n}) \\ (T^*x_2,x_1) & (|T^*|^{2\beta}x_2,x_2) & \ldots & (T^*x_2,x_{2n-1}) & (|T^*|^{2\beta}x_2,x_{2n}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (|T|^{2\alpha}x_{2n-1},x_1) & (Tx_{2n-1},x_2) & \ldots & (|T|^{2\alpha}x_{2n-1},x_{2n-1}) & (Tx_{2n-1},x_{2n}) \\ (T^*x_{2n},x_1) & (|T^*|^{2\beta}x_{2n},x_2) & \ldots & (T^*x_{2n},x_{2n-1}) & (|T^*|^{2\beta}x_{2n},x_{2n}) \end{vmatrix}.$$

Then

$$(\mathrm{I}_2) \qquad 0 \le G_{2n} \le \prod_{j=1}^{n} \||T|^{\alpha}x_{2j-1}\|^2 \||T^*|^{\beta}x_{2j}\|^2.$$

Let $S_1(\alpha)$ and $S_2(\beta)$ be two systems of vectors defined by

$$S_1(\alpha)=\{|T|^{2\alpha}x_1, T^*x_2, |T|^{2\alpha}x_3, T^*x_4, \ldots, |T|^{2\alpha}x_{2n-1}, T^*x_{2n}\}$$

and

$$S_2(\beta)=\{Tx_1, |T^*|^{2\beta}x_2, Tx_3, |T^*|^{2\beta}x_4, \ldots, Tx_{2n-1}, |T^*|^{2\beta}x_{2n}\}.$$

(1) $0<\alpha<1$. (i) *$G_{2n}=0$ iff $S_1(\alpha)$ is a system of linearly dependent vectors iff $S_2(\beta)$ is a system of linearly dependent vectors.*

(ii) *The following conditions* (a) *and* (b) *are equivalent:*
- (a) *the right-hand side equality in* (I_2) *holds,*
- (b) *the following* (b_1) *or* (b_2) *holds:*
- (b_1) $(|T^*|^{2\beta}x_{2j},x_{2k})=0$ *for* $j\ne k$, $(|T|^{2\alpha}x_{2j-1},x_{2k-1})=0$ *for* $j\ne k$ *and* $(Tx_{2j-1},x_{2k})=0$ *for* $j,k=1,2,\ldots,n$.
- (b_2) $0\in S_1(\alpha)$ (*equivalently,* $0\in S_2(\beta)$).

(2) $\alpha=1$. (i) *$G_{2n}=0$ iff $S_2(0)=\{Tx_1,x_2,Tx_3,x_4,\ldots,Tx_{2n-1},x_{2n}\}$ is a system of linearly dependent vectors.*

(ii) *The following conditions* (a) *and* (b) *are equivalent:*
- (a) *the right-hand side equality in* (I_2) *holds,*
- (b) *the following* (b_1) *or* (b_2) *holds:*
- (b_1) $(x_{2j},x_{2k})=0$ *for* $j\ne k$, $(|T|^2x_{2j-1},x_{2k-1})=0$ *for* $j\ne k$ *and* $(Tx_{2j-1},x_{2k})=0$ *for* $j,k=1,2,\ldots,n$.
- (b_2) $0\in S_2(0)$.

(3) $\alpha=0$. (i) *$G_{2n}=0$ iff $S_1(0)=\{x_1,T^*x_2,x_3,T^*x_4,\ldots,x_{2n-1},T^*x_{2n}\}$ is a system of linearly dependent vectors.*

(ii) *The following conditions* (a) *and* (b) *are equivalent:*
- (a) *the right-hand side equality in* (I_2) *holds,*
- (b) *the following* (b_1) *or* (b_2) *holds:*
- (b_1) $(|T^*|^2x_{2j},x_{2k})=0$ *for* $j\ne k$, $(x_{2j-1},x_{2k-1})=0$ *for* $j\ne k$ *and* $(x_{2j-1},T^*x_{2k})=0$ *for* $j,k=1,2,\ldots,n$.
- (b_2) $0\in S_1(0)$.

COROLLARY 1. *For any operator T on H,*

$$(\mathrm{I}_3) \qquad |(Tx,y)|^2 \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)$$

holds for any x,y in H and any real number α with $0\le\alpha\le 1$.

(1) $0<\alpha<1$. *The equality in* (I_3) *holds iff* $|T|^{2\alpha}x$ *and* T^*y *are linearly dependent iff* Tx *and* $|T^*|^{2(1-\alpha)}y$ *are linearly dependent.*

(2) $\alpha=1$. *The equality in* (I_3) *holds iff* Tx *and* y *are linearly dependent.*

(3) $\alpha=0$. *The equality in* (I_3) *holds iff* x *and* T^*y *are linearly dependent.*

REMARK 1. (I_3) is obtained in [**1**, **8**, **9**] without the proof of the case when the equality in (I_3) holds. As (I_3) is equivalent to the following inequality (I_4): $|(Tx,y)|\leq \||T|^{\alpha}x\|\||T^*|^{1-\alpha}y\|$, so that one might believe that the equality in (I_3) or (I_4) would hold iff $|T|^{2\alpha}x$ and $|T^*|^{2(1-\alpha)}y$ are linearly dependent or $|T|^{\alpha}x$ and $|T^*|^{1-\alpha}y$ are linearly dependent. But here we can give a simple counterexample as follows. In case $\alpha=\frac{1}{2}$, let $T=\left(\begin{smallmatrix}0&2\\1&0\end{smallmatrix}\right)$, $x=\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $y=\left(\begin{smallmatrix}1\\4\end{smallmatrix}\right)$; then $|T^*|y=\left(\begin{smallmatrix}2\\4\end{smallmatrix}\right)=2|T|x$, namely $|T|x$ and $|T^*|y$ are linearly dependent, but $|(Tx,y)|^2=36\neq(|T|x,x)(|T^*|y,y)=54$. Also we can give an example such that the equality in (I_3) or (I_4) does not hold even if $|T|x$ and $|T^*|^{1-\alpha}y$ are linearly dependent. We would like to emphasize that the equality in (I_3) or (I_4) holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent iff Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent in case α with $0<\alpha<1$. Corollary 1 is shown in [**4**].

REMARK 2. Put $\alpha=\frac{1}{2}$ in Corollary 1; then we have the useful inequality $|(Tx,y)|^2\leq(|T|x,x)(|T^*|y,y)$, the so-called "mixed Schwarz inequality" [**6**] as an extension of $|(Tx,y)|^2\leq(Tx,x)(Ty,y)$ for positive operator T. By Corollary 1, the equality in this mixed Schwarz inequality holds iff $|T|x$ and T^*y are linearly dependent iff Tx and $|T^*|y$ are linearly dependent.

3. Proofs of the results. In order to show the results, we need the following

THEOREM A. *For any* $x_1,x_2,\dots,x_n\in H$, *let* G_n *be the determinant of a square matrix of order* n *defined by* $G_n=|((x_j,x_k))|$. *Then*

$$0\leq G_n\leq\|x_1\|^2\|x_2\|^2\cdots\|x_n\|^2.$$

On the left-hand side, equality holds if and only if $x_1,x_2,\dots,x_n$ *are linearly dependent. On the right-hand side, equality holds if and only if* $x_1,x_2,\dots,x_n$ *are mutual orthogonal or* $\{x_1,x_2,\dots,x_n\}$ *contains the zero vector.*

The right-hand side inequality in Theorem A is Hadamard's theorem, and the left-hand side inequality in Theorem A is well known and can be considered as a generalization of Schwarz inequality. Many ingenious and elegant proofs of Hadamard's theorem have been given by many authors (for example, [**2**, **5**, **7**, **11**]).

LEMMA. *Let* $T=U|T|$ *be the polar decomposition of* T, *where* U *means the partial isometry and* $|T|=(T^*T)^{1/2}$ *with* $N(U)=N(|T|)$, *where* $N(S)$ *denotes the kernel of an operator* S. *Then, for any positive number* p,

$$|T^*|^p=U|T|^pU^*. \tag{i}$$

PROOF. First of all, we state the following obvious but important relation $(*)$:

$(*)$ $N(S^q)=N(S)$ for any positive operator S and for any positive number q.

As U^*U is the initial projection, then we have $U^*U|T|^q=|T|^q$ for any positive number q because the relation $(*)$ for $|T|$ yields $\overline{R(|T|^q)}=\overline{R(|T|)}$. Then

$$|T^*|^2=TT^*=U|T|\,|T|U^*=U|T|U^*U|T|U^*=(U|T|U^*)^2,$$

so $|T^*| = U|T|U^*$ since $U|T|U^*$ is positive. By induction, we have $|T^*|^{n/m} = U|T|^{n/m}U^*$ for any natural number m and n because $U^*U|T|^q = |T|^q$; then the continuity of T yields $|T^*|^p = U|T|^pU^*$ by attending $n/m \to p$, so the proof of (i) is complete.

PROOF OF THEOREM 1. (1) $0<\alpha<1$. In Theorem A, we replace x_1 by $|T|^\alpha x_1$, and x_k by $|T|^\beta U^* x_k$ for $k = 2, 3, \ldots, n$, and for any positive number α and β with $\alpha + \beta = 1$. Then, by the Lemma we have the following:

$$(|T|^\alpha x_1, |T|^\beta U^* x_k) = (U|T|x_1, x_k) = (Tx_1, x_k) \quad \text{for } k = 2, 3, \ldots, n,$$
$$(|T|^\beta U^* x_j, |T|^\beta U^* x_k) = (U|T|^{2\beta} U^* x_j, x_k) = (|T^*|^{2\beta} x_j, x_k) \quad \text{for } j, k = 2, 3, \ldots, n.$$

By Theorem A and the Lemma, we have

$$\begin{aligned} 0 \le G_n &\le \||T|^\alpha x_1\|^2 \||T|^\beta U^* x_2\|^2 \||T|^\beta U^* x_3\|^2 \cdots \||T|^\beta U^* x_n\|^2 \\ &= \||T|^\alpha x_1\|^2 \||T^*|^\beta x_2\|^2 \||T^*|^\beta x_3\|^2 \cdots \||T^*|^\beta x_n\|^2. \end{aligned}$$

$G_n = 0$ iff $|T|^\alpha x_1$, $|T|^\beta U^* x_2$, $|T|^\beta U^* x_3, \ldots, |T|^\beta U^* x_n$ are linearly dependent by Theorem A iff $|T|^{2\alpha} x_1$, $|T|U^* x_2$, $|T|U^* x_3, \ldots, |T|U^* x_n$ are linearly dependent by the relation $(*)$ for $|T|$ iff (D_1): $S_1(\alpha) = \{|T|^{2\alpha} x_1, T^* x_2, T^* x_3, \ldots, T^* x_n\}$ is a system of linearly dependent vectors.

On the other hand, $G_n = 0$ iff $|T|^\alpha x_1$, $|T|^\beta U^* x_2, |T|^\beta U^* x_3, \ldots, |T|^\beta U^* x_n$ are linearly dependent iff $|T|x_1$, $|T|^{2\beta} U^* x_2$, $|T|^{2\beta} U^* x_3, \ldots, |T|^{2\beta} U^* x_n$ are linearly dependent by the relation $(*)$ for $|T|$, equivalently, $U|T|x_1$, $U|T|^{2\beta} U^* x_2$, $U|T|^{2\beta} U^* x_3, \ldots,$ $U|T|^{2\beta} U^* x_n$ are linearly dependent by $(*)$ for $|T|$ and $N(U) = N(|T|)$ iff (D_2): $S_2(\beta) = \{Tx_1, |T^*|^{2\beta} x_2, |T^*|^{2\beta} x_3, \ldots, |T^*|^{2\beta} x_n\}$ is a system of linearly dependent vectors by (i) of the Lemma. Therefore, (D_1) holds iff (D_2) holds.

The proof of equality for the right-hand side of (I_1) follows by Theorem A and the argument stated above in the first half of the proof, so the proof of the case (1) is complete. The proofs of the case (2) $\alpha = 1$ and (3) $\alpha = 0$ are obvious, so we omit them. Hence we have finished the proof of Theorem 1.

PROOF OF THEOREM 2. (1) $0 < \alpha < 1$. In Theorem A, we replace x_{2k} by $|T|^\beta U^* x_{2k}$ for $k = 1, 2, \ldots, n$, and x_{2k-1} by $|T|^\alpha x_{2k-1}$ for $k = 1, 2, \ldots, n$, and for any positive number α and β with $\alpha + \beta = 1$. Then, by the Lemma we have the following:

$$(|T|^\beta U^* x_{2j}, |T|^\beta U^* x_{2k} = (U|T|^{2\beta} U^* x_{2j}, x_{2k}) = (|T^*|^{2\beta} x_{2j}, x_{2k}),$$
$$(|T|^\alpha x_{2j-1}, |T|^\beta U^* x_{2k}) = (U|T|x_{2j-1}, x_{2k}) = (Tx_{2j-1}, x_{2k})$$

for $j, k = 1, 2, \ldots, n$. By Theorem A and the Lemma, we have

$$\begin{aligned} 0 \le G_{2n} &\le \||T|^\alpha x_1\|^2 \||T|^\beta U^* x_2\|^2 \||T|^\alpha x_3\|^2 \||T|^\beta U^* x_4\|^2 \\ &\qquad \cdots \||T|^\alpha x_{2n-1}\|^2 \||T|^\beta U^* x_{2n}\|^2 \\ &= \||T|^\alpha x_1\|^2 \||T^*|^\beta x_2\|^2 \||T|^\alpha x_3\|^2 \||T^*|^\beta x_4\|^2 \cdots \||T|^\alpha x_{2n-1}\|^2 \||T^*|^\beta x_{2n}\|^2. \end{aligned}$$

The proofs of the left-hand side and right-hand side of the equality in (I_2) are given by the same way as in the proofs of Theorem 1, so we omit them. The proofs of the case $\alpha = 1$ and $\alpha = 0$ are obvious, so we omit them. Hence we have finished the proof of Theorem 2.

PROOF OF COROLLARY 1. The proof follows by the left-hand side inequality of Theorem 1 or Theorem 2.

Other generalizations of the Heinz inequality were obtained in [**1** and **10**]. Results in this paper would remain valid for unbounded operators under slight modifications.

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REFERENCES

1. J. Dixmier, *Sur une inegalité de E. Heinz*, Math. Ann. **126** (1953), 75–78.
2. T. Furuta, *An elementary proof of Hadamard's theorem*, Mat. Vestnik **23** (1971), 267–269.
3. ——, *Mixed Hadamard's theorems*, Proc. Amer. Math. Soc. **96** (1986), 217–220.
4. ——, *A simplified proof of Heinz inequality and scrutiny of its equality*, Proc. Amer. Math. Soc. **97** (1986), 751–753.
5. F. G. Gantmacher, *The theory of matrices*, Vol. 1, Chelsea, New York, 1960.
6. P. Halmos, *A Hilbert space problem book*, 2nd ed., Springer-Verlag, Berlin and New York, 1982.
7. G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press, 1934.
8. E. Heinz, *On an inequality for linear operators in a Hilbert space*, Report on Operator Theory and Group Representations, no. 387, National Academy of Science, U.S.A., 1955, pp. 27–29.
9. T. Kato, *Note on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.
10. ——, *A generalization of Heinz inequality*, Proc. Japan Acad. **37** (1961), 305–308.
11. H. Widom, *Lectures on integral equations*, Van Nostrand, 1969.

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