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WEAK-STAR GENERATORS OF Z^n, $n \geq 1$, AND TRANSITIVE OPERATOR ALGEBRAS

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ABSTRACT. A function f in H^∞ is said to be a weak-star generator (w*-gen.) of the function $e_n(z) = z^n$, $|z| < 1$, $n \geq 1$, if $\lim_\alpha p_\alpha \circ f = e_n$ (w*-topology), for some net (p_α) of complex polynomials. For the case $n = 1$, f is called a w*-gen. of H^∞. The w*-generators of H^∞ have been defined and characterized by Sarason. It is the purpose of the present paper to give necessary and sufficient conditions for a function to generate e_n. As a result, it follows from our characterization that certain analytic Toeplitz operators have the transitive algebra property.

1. Introduction. Let H^∞ denote the algebra of bounded analytic functions in the open unit disk $\mathbf{D} = \{z\colon |z| < 1\}$. This algebra is naturally identified with the subalgebra $H^\infty(\mathbf{T}) = \{f\colon \hat{f}(j) = 0,\ j = -1, -2, \ldots\}$, of $L^\infty = L^\infty(\mathbf{T})$, where the unit circle $\mathbf{T} = \{z\colon |z| = 1\}$ is endowed with the normalized Lebesgue measure. The algebra $H^\infty(\mathbf{T})$ is the dual of a quotient space of $L^1(\mathbf{T})$ and as such has a w*-topology. We will refer to this topology as the w*-topology of H^∞. A function φ in H^∞ is said to be a w*-generator if the complex polynomials in φ are w*-dense in H^∞. Sarason presented this definition in **[5]** and later characterized the w*-generators of H^∞ **[6]**.

In the present paper, we will extend Sarason's result by introducing and characterizing weak-star generators of the functions $e_n(z) = z^n$, $z \in \mathbf{D}$, $n \geq 1$. We will also give an application of our result to operator theory by showing certain analytic Toeplitz operators have the transitive algebra property. We will conclude by listing open questions which arise from our work.

2. Main results. For each $n \geq 1$, we let

$$B_n(z) = \prod_{i=1}^{n} (z - \xi_i)(1 - \overline{\xi}_i z)^{-1}, \qquad |\xi_i| < 1,$$

be a finite Blaschke product of the open unit disk $\mathbf{D}$. We let e_n denote the special case of B_n, when $\xi_i = 0$, $1 \leq i \leq n$.

Throughout, let $\mathscr{P}$ denote the set of all complex polynomials.

DEFINITION 2.1. A function $f \in H^\infty$, is said to be a w*-gen. of the function B_n, if, there exists a net $(P_\alpha) \subset \mathscr{P}$ such that $\lim_\alpha \mathscr{P}_\alpha \circ f = B_n$ (w*-topology).

DEFINITION 2.2 (SARASON). A w*-gen. of the function e_1 is said to be a w*-gen. of H^∞ **[5]**.

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THEOREM 2.1. *If f is a w*-gen. of H^∞ and if k is a positive divisor of n, then the function $f_k = f \circ e_k$ is a w*-gen. of e_k, and thus a w*-gen. of e_n.*

PROOF. Let $h \in L^1 = L^1(\mathbf{T})$ and define functions $h_m \in L^1$, $1 \le m \le k$, by

$$h_m(e^{it}) = h(e^{it_m}), \qquad \text{where } t_m = [t + 2(m-1)\pi]/k.$$

For every $p \in \mathscr{P}$ we have

(1)

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}(p\circ f_k - e_k)(e^{it})h(e^{it})\,dt &= \frac{1}{2k\pi}\int_0^{2k\pi}(p\circ f - e_1)(e^{it})h(e^{it/k})\,dt \\ &= \sum_{m=1}^{k}\frac{1}{2k\pi}\int_{2(m-1)\pi}^{2m\pi}(p\circ f - e_1)(e^{it})h(e^{it/k})\,dt \\ &= \sum_{m=1}^{k}\frac{1}{2k\pi}\int_0^{2\pi}(p\circ f - e_1)(e^{it})h_m(e^{it})\,dt.\end{aligned}$$

The fact that f is a w*-gen. of H^∞ together with (1) implies that f_k is a w*-gen. of e_k; and thus f_k is a w*-gen. of e_n. $\square$

Recall that if G is a bounded, open, connected, and simply connected subset of the complex plane, $H^\infty(G)$ denotes the algebra of all bounded analytic functions on G and

$$\|f\|_\infty = \sup_{z\in G}|f(z)|, \qquad f \in H^\infty(G).$$

Also, recall that a subset G_0 of G is said to be dominating for G if

$$\sup_{z\in G_0}|f(z)| = \sup_{z\in G}|f(x)|,$$

for all $f \in H^\infty(G)$.

In order to show that the sufficient condition given by Thoerem 2.1 is also necessary, we will need the following result of Sarason together with the fact that w*-gen. of H^∞ are necessarily univalent [**5**, Proposition 3].

THEOREM 2.2 [**6**]. *Let $f \in H^\infty$ be univalent and let $G_0 = f(\mathbf{D})$. The function f is not a w*-gen. of H^∞, if and only if, there exists a bounded, open, connected, and simply connected set G such that $G_0 \subsetneqq G$, G_0 is dominating for G. In this case G can be chosen to have the following additional property.*

$$(\dagger) \qquad (H^\infty(G)|G_0)\circ f = w^*\text{-closure}\,\{p\circ f : p \in \mathscr{P}\}.$$

REMARK. The condition (†) does not appear in [**6**]. However, we will indicate how (†) will follow from some of the proofs and results of Sarason's paper.

Let M^α be the w*-closure of polynomials in the function f and let γ be the order of G_0, where α and γ are countable ordinal numbers. For the details consult [**6**, pp. 521 and 525].

If $\alpha \ge \gamma$, then $G_0^\alpha = G_0^\gamma$ and therefore by Theorem 2 of [**6**] we have

$$M^\alpha = (H^\infty(G_0^\alpha)|G_0)\circ f.$$

If $\alpha < \gamma$, then $M^\alpha = M^\gamma$ and by the same theorem we have

$$M^\gamma = (H^\infty(G_0^\gamma)|G_0)\circ f.$$

Thus, if we let $G = G_0^\gamma$, G would have the stated properties and (†) is satisfied. See the proof of Corollary 2 of [**6**].

Theorem 2.3 and 2.4 will imply that the sufficient condition of Theorem 2.1 is also necessary.

THEOREM 2.3. *If f is a w*-gen. of e_n and if f is univalent, then f is a w*-gen. of H^∞.*

PROOF. We will prove the theorem by assumsing that f is *not* a w*-gen. of H^∞ and arriving at a contradiction. To this end, the function f satisfies the hypothesis of Theorem 2.2 and hence there exists an open, bounded, connected, and simply connected subset G of the complex plane such that

$$(1) \qquad f(\mathbf{D}) = G_0 \subsetneqq G, \qquad (H^\infty(G)|G_0) \circ f = \text{w}^*\text{-clos.}\{p \circ f : p \in \mathscr{P}\},$$

and G_0 is dominating for G.

Define the function $g_0 : G_0 \to \mathbf{D}$ by

$$(2) \qquad g_0(w) = [f^{-1}(w)]^n, \quad \text{for } w = f(z) \in G_0.$$

The fact that f is a w*-gen. of e_n together with (1) and (2) implies that there exists a function $g \in H^\infty$ such that $g_0 = g|G_0$. Since G_0 is dominating for G and $|g_0(w)| \le 1$ for all $w \in G_0$, we have

$$(3) \qquad |g(w)| \le 1, \quad \text{for all } w \in G.$$

Since $G_0 \ne G$, there exists a point $w_0 \in (\partial G_0) \cap G$. Hence, there exists a sequence $(w_m) \subset G_0$ converging to w_0 and will show that $\lim_m |f^{-1}(w_m)| = 1$. To this end, assume to the contrary and choose a subsequence w_k' of (w_m) and a point $z' \in \mathbf{D}$ such that $\lim_k f^{-1}(w_k') = z'$. Thus, $f(z') = \lim_k f(f^{-1}(w_k')) = \lim_k w_k' = w_0 \notin G_0$. Therefore, we must have

$$\lim_m |f^{-1}(w_m)| = 1.$$

Thus,

$$(4) \qquad \begin{aligned} |g(w_0)| &= \lim_m |g(w_m)| = \lim_m |g_0(w_m)| \\ &= \lim_m |[f^{-1}(w_m)]^n| = 1. \end{aligned}$$

If we let $g(w_0) = z_0$, we then conclude from (4) that $|z_0| = 1$. Since g is a nonconstant analytic function defined in G, it follows that g maps a neighborhood of w_0 onto a neighborhood of z_0. Since $|z_0| = 1$, there exists a point $w_0' \in G - \{w_0\}$, such that w_0' is sufficiently close to the point w_0 and $|g(w_0')| > 1$.

By (3) we now reach our final contradiction, and that the univalent function f must be a w*-gen. of H^∞. □

We will now show that the sufficient condition given by Theorem 2.1 is also necessary even if f is not univalent.

THEOREM 2.4. *If f is a w*-gen. of e_n and if f is not univalent, then there exist a w*-gen. g of H^∞ and a positive divisor k of n such that $f = g \circ e_k$.*

PROOF. We will present the proof for the cases $n = 4$; the general case can be proved in a similar manner.

To this end, let $z_1 \neq z_2$ be two points in $\mathbf{D}$ such that $f(z_1) = fd(z_2)$. It now follows from the hypothesis that $z_1^4 = z_2^4$ and hence $z_1 = \delta z_2$ for some $\delta \in H$, where $H = \{1, -1, i, -i\}$ is the multiplicative group of fourth roots of unity.

Recall that the order of δ is the smallest positive integer m such that $\delta^m = 1$. Let

$$k = \max\{\mathrm{ord}(\delta) | z_1 = \delta z_2, z_1 \neq z_2, \text{ and } f(z_1) = f(z_2)\}.$$

Note that either $k = 2$ or $k = 4$.

If $k = 2$, then $\delta = -1$ and $z_1 = -z_2$. Define the set

$$S = \{z \in \mathbf{D} \colon f(z) = fd(-z)\}$$

and observe that S is a nonempty relatively closed subset of $\mathbf{D}$. We will show that $S = \mathbf{D}$ by showing that S is also relatively open. To this end, let $z_0 \in S$ and let U_1, U_2 be neighborhoods of $z_0, -z_0$ respectively such that:

(i) $U_1 \cap U_2 = (iU_1) \cap U_2 = (-iU_1) \cap U_2 \neq \varnothing$; and

(ii) $f(U_1) = f(U_2)$.

If $z \in U_1$, then by (ii) there exists a point $z' \in U_2$ such that $f(z) = f(z')$; and hence $z^4 = z'^4$. It is now evident from (i) that $z = -z'$. Thus $U_1 \subseteq S$, and S is relatively open in $\mathbf{D}$. Therefore, $S = \mathbf{D}$ and there exists a function $g \in H^\infty$ such that $f = g \circ e_2$.

We will now prove that g is univalent. Assume to the contrary that there are distinct complex numbers $z_1, z_2 \in \mathbf{D}$ such that

$$g(z_1) = g(z_2). \tag{1}$$

If w_j is a square root of z_j, we then have

$$f(w_j) = (g \circ e_2)(w_j) = g(w_j^2) = g(z_j), \qquad j = 1, 2. \tag{2}$$

It follows from (1) and (2) that $f(w_1) = f(w_2)$. Since $e_4 \in$ w*-closure $\mathscr{P}_\alpha \circ f$ and $f(w_1) = f(w_2)$, we conclude that $w_1^4 = w_2^4$. By the maximality of k we must have $w_1 = -w_2$; thus $z_1 = z_2$. This contradicts the fact that z_1 and z_2 were chosen to be distinct. Hence, g must be univalent.

If $k = 4$, then there is a pair of distinct points z_1 and z_2 such that $f(z_1) = f(z_2)$ and $z_1 = \delta z_2$, where $\delta = i$ or $\delta = -i$. For definiteness assume that $\delta = i$ and define the set

$$S' = \{z \in \mathbf{D} | f(z) = f(iz)\}.$$

Just as in the previous case, one shows that $S' = \mathbf{D}$ and that $f(z) = f(iz)$ for all $z \in \mathbf{D}$. Hence, there exists a function $h \in H^\infty$ such that $f = h \circ e_4$ and it is easy to show that h is univalent.

By adopting an argument similar to the proof of Theorem 2.3, one can show that the functions g and h are indeed w*-gen. of H^∞. Thus, in either case, $k = 2$ or $k = 4$, the theorem is proven. □

3. Application to operator theory. An algebra $\mathscr{U}$ of operators (linear and bounded) on a separable, infinite dimensional, and complex Hilbert space. $\mathscr{H}$ is transitive if $\mathscr{U}$ has only trivial invariant subspaces. An operator $T \colon \mathscr{H} \to \mathscr{H}$ is said to have the transitive algebra property if whenever $\mathscr{U}$ is a transitive operator algebra and $T \in \mathscr{U}$, then $\mathscr{U}$ is weakly dense in $\mathscr{L}(\mathscr{H})$, the algelbra of all operators. There are several classes of operators which are known to have the transitive algebra property. For a survey of results see [**1**, **2**, **3** and **4**, Chapter 8].

Recall the Hardy space H^2 as the class of all functions f analytic in the disk $\mathbf{D}$ such that

$$f(z) = \sum_{j\geq 0} \hat{f}(j)z^j, \quad z \in \mathbf{D}; \qquad \|f\|^2 \overset{\text{def}}{=} \sum_{j\geq 0} |\hat{f}(j)|^2 < \infty.$$

Also, recall the analytic Toeplitz operator $T_\varphi \colon H^2 \to H^2$ with the symbol $\varphi \in H^\infty$ is the multiplication operator defined by

$$(T_\varphi f)(z) = \varphi(z)f(z), \qquad f \in H^2.$$

If the symbol $\varphi = e_n$, then T_φ is called a unilateral shift of multiplicity n. Unilateral shifts of finite multiplicity are known to have the transitive algebra property [**4**, Theorem 8.18].

THEOREM 3.1. *If $\varphi \in H^\infty$ is a w^*-gen. of the function B_n, then the analytic Toeplitz operator T_φ has the transitive algebra property.*

PROOF. Let $\mathscr{U}$ be a transitive operator algebra containing T_φ. It follows from the hypothesis that

$$\lim_\alpha p_\alpha \circ \varphi = B_n \quad (\text{w}^*\text{-topology}).$$

If $f \in H^2$, then the function $h(e^{it}) = |f(e^{it})|^2$ is in L^1 and we therefore have

$$\lim_\alpha \frac{1}{2\pi} \int_0^{2\pi} (p_\alpha \circ \varphi - B_n)(e^{it})h(e^{it})\,dt = 0. \tag{1}$$

It now follows from (1) that

$$\lim_\alpha T_{p_\alpha \circ \varphi} = T_{B_n} \quad (\text{weak operator topology}).$$

Since the operator $T_{p_\alpha \circ \varphi} = p_\alpha(T_\varphi) \in \mathscr{U}$, it follows that the operator T_{B_n} lies in the weak operator topology closure of $\mathscr{U}$. It is easy to show that the operator T_{B_n} is unitarily equivalent to the shift of multiplicity n, thus $\mathscr{U}$ is weakly dense in $\mathscr{L}(H^2)$, by [**4**, Theorem 8.18]. □

COROLLARY 3.2. *If φ is a w^*-gen. of e_n, then the operator $T_\varphi \oplus T_\varphi \in \mathscr{L}(H^2 \oplus H^2)$ has the transitive algebra property.*

PROOF. It is easy to show that the operator $T_\varphi \oplus T_\varphi$ is unitarily equivalent to the analytic Toeplitz operator $T_{\varphi \circ e_2}$. Since, the function $\varphi \circ e_2$ is a w*-gen. of e_{2n}, it follows from Theorem 3.1 that $T_{\varphi \circ e_2}$ has the transitive algebra property. □

OPEN QUESTIONS. Theorem 3.1 would suggest the following question:

QUESTION 1. What are the w*-generators of the function B_n?

An affirmative answer to the following question, would extend Corollary 3.2.

QUESTION 2. If the functions φ and ψ are two distinct w*-generators of the function e_n, must the operator $T_\varphi \oplus T_\psi \in \mathscr{L}(\mathscr{H}^2 \oplus H^2)$, have the transitive algebra property?

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