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DEGREES OF CONSTANT-TO-ONE FACTOR MAPS

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ABSTRACT. Let f be a constant-to-one endomorphism of degree d, of a subshift of finite type Σ_A. If p is a prime dividing d, then p divides every nonleading coefficient of χ_A, the characteristic polynomial for A. Further constraints are given for the possible degrees of a constant-to-one factor map between subshifts of finite type.

0. Introduction. It is well known that if Σ_A and Σ_B are irreducible subshifts of finite type of equal entropy, and $f\colon \Sigma_A \to \Sigma_B$ is a factor map (that is, a continuous, surjective map which commutes with the shift), then there is a positive integer d such that f is d-to-1 almost everywhere (i.e., every bilaterally transitive point of Σ_B has exactly d preimages under f). The integer d is called the *degree* of f. The degree cannot be arbitrary: for a given Σ_A and Σ_B, there are algebraic constraints on the possible degrees of factor maps $\Sigma_A \to \Sigma_B$. The first such constraint was given by L. R. Welch [14.9 of **H**], who showed that if f is an endomorphism of the full shift on n symbols, then the degree of f divides a power of n. More recently, Boyle [**B1**] has shown that if S and T are sofic systems of entropy $\log\lambda$, then there exists a finite set E of positive integers such that if $f\colon S \to T$ has degree d, then $d = eu$ where $e \in E$ and u is a unit in $\mathbf{Z}[1/\lambda]$. In this paper we give an analogous result for constant-to-one maps between subshifts of finite type. (f is *constant-to-one* if every point has exactly d preimages.) More specifically, we show that if Σ_A and Σ_B are subshifts of finite type and λ is *any* nonzero eigenvalue for A and B (and λ has the same multiplicity in χ_A and χ_B), then there exists a finite set E such that for any constant-to-one map $f\colon \Sigma_A \to \Sigma_B$ of degree d, we have $d = eu$, where $e \in E$ and u is a unit in $\mathbf{Z}[1/\lambda]$. As in Boyle's paper, this implies that if f is an endomorphism, then in fact d must be a unit in $\mathbf{Z}[1/\lambda]$. It follows that if p is a prime dividing d, then p divides every nonleading coefficient of χ_A.

In [**B1**], Boyle proves that if $f\colon S \to T$ (S, T sofic, of entropy $\log\lambda$) has degree d, then the inverse image of certain cylinder sets breaks up into d sets of equal measure, and so the measure of these sets is divisible by d in $\mathbf{Z}[1/\lambda]$. In this paper, we show that under our more restrictive conditions, the inner product of a generalized left eigenvector and a right eigenvector for the matrix A is divisible by d in $\mathbf{Z}[1/\lambda]$, for any nonzero eigenvalue λ. The remainder of the proof then follows exactly as in [**B1**].

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1. Background. We will assume some familiarity with subshifts of finite type and symbolic dynamics. For further background, see [**A-M**] and [**P-T**].

For the remainder of this paper, we will assume that Σ_A and Σ_B are irreducible subshifts of finite type, where A and B are nonnegative, integral matrices. Let $\mathscr{S}_A$ denote the symbol set of Σ_A. Thus $\mathscr{S}_A$ is the collection of edges in the directed graph for A and $\Sigma_A = \{x \in \mathscr{S}_A^{\mathbf{Z}} : x_{i+1} \text{ follows } x_i \text{ for all } i\}$.

A one block map $f\colon \Sigma_A \to \Sigma_B$ is called *right resolving* if whenever $x_1 \in \mathscr{S}_A$, $y_1, y_2 \in \mathscr{S}_B$, $f(x_1) = y_1$ and $y_1 y_2$ is allowed, then there exists a unique $x_2 \in \mathscr{S}_A$ such that $f(x_2) = y_2$ and $x_1 x_2$ is allowed. f is *left resolving* if whenever $x_2 \in \mathscr{S}_A$, $y_1, y_2 \in \mathscr{S}_B$, $f(x_2) = y_2$ and $y_1 y_2$ is allowed, then there exists a unique $x_1 \in \mathscr{S}_A$ with $f(x_1) = y_1$ and $x_1 x_2$ allowed.

A one block map $f\colon \Sigma_A \to \Sigma_B$ (i.e. a map on edges) defines a map $\hat{f}$ on vertices of the directed graph of A as follows: if i is a vertex and e is any edge beginning at i, then $\hat{f}(i)$ is the initial vertex of $f(e)$. Clearly $\hat{f}$ is well defined.

If A is $n \times n$ and B is $k \times k$, then the *relation matrix* F for f is defined by

$$R_{ij} = \begin{cases} 1 & \text{if } \hat{f}(i) = j, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \le i \le n$, $1 \le j \le k$. It is easy to check that if f is right resolving then $AR = RB$, and if f is left resolving then $R^T A = BR^T$ (T denotes transpose).

We will use the following theorem.

THEOREM (NASU [**N**, COROLLARY 6.4]; ALSO IMPLICIT IN [**K**]). *Let Σ_A and Σ_B be irreducible subshifts of finite type, of equal entropy, and assume $f\colon \Sigma_A \to \Sigma_B$ is constant-to-one. Then there exist topological conjugacies $g\colon \Sigma_A \to \Sigma_{\overline{A}}$ and $h\colon \Sigma_B \to \Sigma_{\overline{B}}$, and a one-block map $\overline{f}\colon \Sigma_{\overline{A}} \to \Sigma_{\overline{B}}$ which is right and left resolving and such that $f = h^{-1}\overline{f}g$.*

It is easy to see that if f is constant-to-one, of degree d, then $\overline{f}$ must be exactly d-to-one as a map on symbols.

2. Main results. In this section, we prove our main theorem, which should be compared with [**B1**, Theorem 2.2 and Corollary 2.4].

THEOREM 1. *Let Σ_A be an irreducible subshift of finite type, and λ a nonzero eigenvalue for A. Then there is a finite set E of positive integers such that if Σ_B is an irreducible subshift of finite type, and λ is an eigenvalue for B which has the same multiplicity in χ_A and χ_B, and if $f\colon \Sigma_A \to \Sigma_B$ is a constant-to-one factor map of degree d, then $d = eu$, where $e \in E$ and u is a unit in $\mathbf{Z}[1/\lambda]$.*

COROLLARY. *Let f be a constant-to-one endomorphism of Σ_A and λ a nonzero eigenvalue. If f has degree d, then d is a unit in $\mathbf{Z}[1/\lambda]$.*

Later we will show that Theorem 1 is a consequence of the following:

THEOREM 2. *Assume the same hypotheses as in Theorem 1. Let*

$$r = (r_1, \dots, r_n)^T, \qquad r_i \in \mathbf{Z}[1/\lambda],$$

be a right eigenvector for A, and $l = (l_1, \dots, l_n)$, $l_i \in \mathbf{Z}[1/\lambda]$, a generalized left eigenvector (i.e. $l(A - \lambda I)^s = 0$ for some s). If $f\colon \Sigma_A \to \Sigma_B$ is a constant-to-one factor map of degree d, then d divides $l \cdot r$ in $\mathbf{Z}[1/\lambda]$.

REMARK. Such an l and r always exist—we get $l_i, r_i \in \mathbf{Q}[\lambda]$ by linear algebra, and by multiplying l and r by a sufficiently large integer, we force l_i, r_i to lie in $\mathbf{Z}[\lambda] \subset \mathbf{Z}[1/\lambda]$ (the last inclusion since χ_A is monic).

First we prove a lemma.

LEMMA. *Let l and r be as in Theorem 2, and let $\Sigma_{\overline{A}}$ be topologically conjugate to Σ_A. Then there exist a right eigenvector $r' = (r'_1, \ldots, r'_m)^T$, $r'_i \in \mathbf{Z}[1/\lambda]$, and a generalized left eigenvector $l' = (l'_1, \ldots, l'_m)$, $l'_i \in \mathbf{Z}[1/\lambda]$, such that $l' \cdot r' = l \cdot r$.*

PROOF. Proceeding inductively, we first assume that A and $\overline{A}$ are strong shift equivalent in one step (see [**W1**] for definitions), so that $A = UV$, $\overline{A} = VU$, for nonnegative, integral matrixes U, V. Let $r' = Vr$ and $l' = lU/\lambda$. Then $l' \cdot r' = lU/\lambda \cdot Vr = lAr/\lambda = l \cdot r$. The proof for an arbitrary conjugate pair Σ_A and $\Sigma_{\overline{A}}$ now follows by induction on the length of the strong shift equivalence between A and $\overline{A}$. □

PROOF OF THEOREM 2. We now assume that $f\colon \Sigma_A \to \Sigma_B$ is constant-to-one of degree d, and that $\overline{f}\colon \Sigma_{\overline{A}} \to \Sigma_{\overline{B}}$ is right and left resolving, with $\Sigma_{\overline{A}}$ conjugate to Σ_A and $\Sigma_{\overline{B}}$ conjugate to Σ_B, as in Nasu's theorem. Let l' and r' be the vectors guaranteed by the previous lemma. We will show that d divides $l' \cdot r'$ (and hence $l \cdot r$) in $\mathbf{Z}[1/\lambda]$.

Let R be the relation matrix for $\overline{f}$, so that $\overline{A}R = R\overline{B}$ and $R^T\overline{A} = \overline{B}R^T$. If $\overline{A}$ is $m \times m$, and $\overline{B}$ is $k \times k$, then R defines an injective linear map $L\colon \mathbf{R}^k \to \mathbf{R}^m$. The matrix equations tell us that the generalized right and left eigenspaces for $\overline{B}$ are carried into the generalized right and left eigenspaces for $\overline{A}$ respectively. Since conjugacy preserves the dimensions of generalized eigenspaces corresponding to nonzero eigenvalues (see [**W2**, Corollary 4.8]), the condition that λ has the same multiplicity in χ_A as in χ_B implies that the $\overline{A}$ generalized eigenspaces for λ have the same dimension as the $\overline{B}$ generalized eigenspaces, and so L restricted to generalized eigenspaces is an isomorphism.

It follows from these remarks that l' and r' are in the image of L. Thus $r' = R\beta$ where β is a $\overline{B}$ generalized right eigenvector for λ. Since $r'_i = \sum_{j=1}^k R_{ij}\beta_j$, it follows that $r'_i = \beta_j$ if $\hat{f}(i) = j$ (where $\hat{f}$ is the induced map on vertices), so that the entries of β lie in $\mathbf{Z}[1/\lambda]$. Similarly, $l' = \alpha R^T$, where α is a $\overline{B}$ generalized left eigenvector whose components lie in $\mathbf{Z}[1/\lambda]$. Since f is constant to one, $\overline{f}$ must be exactly d-to-one as a map on symbols, and it is easy to check that $R^TR = dI$. Thus $l' \cdot r' = \alpha R^T R\beta = d\alpha \cdot \beta$. □

PROOF OF THEOREM 1. We now show that Theorem 1 follows from Theorem 2. First, we show that for any eigenvalue λ for A, there exists a right eigenvector r and a generalized left eigenvector l (both corresponding to λ and having components in $\mathbf{Z}[1/\lambda]$) such that $l \cdot r \neq 0$. Assume that A is $n \times n$, and let r be a right eigenvector corresponding to λ. By the remark following Theorem 2, we may assume that the entries of r lie in $\mathbf{Z}[1/\lambda]$. If l is a generalized left eigenvector corresponding to an eigenvalue $\gamma \neq \lambda$, then $l \cdot r = 0$, by a standard argument. Since the generalized eigenvectors span all of $\mathbf{C}^n$, r cannot be perpendicular to every generalized left eigenvector. If we choose a collection of generalized left eigenvectors, with components in $\mathbf{Q}[1/\lambda]$, which span the generalized left eigenspace corresponding to λ, then at least one of these will have nonzero inner product with r. Again, we can multiply this vector by a sufficiently large integer so that its components lie in $\mathbf{Z}[1/\lambda]$.

The remainder of the proof of Theorem 1 now follows exactly as in the proof of [**B1**, Theorem 2.2]. We repeat Boyle's argument here for completeness.

Let l and r be as above. Since $1/(l\cdot r) \in \mathbf{Q}[1/\lambda]$, there exists a positive integer m such that $m/(l\cdot r) \in \mathbf{Z}[\lambda] \subset \mathbf{Z}[1/\lambda]$. Let E be the set of positive integers dividing m. If $f\colon \Sigma_A \to \Sigma_B$ is constant-to-one, of degree d, then $(l\cdot r)/d \in \mathbf{Z}[1/\lambda]$, by Theorem 2. Thus

$$\frac{m}{d} = \frac{m}{l\cdot r}\cdot\frac{l\cdot r}{d} \in \mathbf{Z}\left[\frac{1}{\lambda}\right].$$

Write $d = eu$, where $e = \gcd(m,d) \in E$. Let $k = m/e$. Then

$$\frac{k}{u} = \frac{ke}{ue} = \frac{m}{d} \in \mathbf{Z}\left[\frac{1}{\lambda}\right],$$

with k and u relatively prime. If p is a prime dividing u, then

$$\frac{k}{p} = \frac{u}{p}\cdot\frac{k}{u} \in \mathbf{Z}\left[\frac{1}{\lambda}\right],$$

and therefore $1/p \in \mathbf{Z}[1/\lambda]$. It follows by multiplying that $1/u \in \mathbf{Z}[1/\lambda]$. □

PROOF OF COROLLARY. To prove the corollary, given Σ_A, let E be the finite set guaranteed by Theorem 1. If f is a constant-to-one endomorphism of Σ_A, of degree d, then $d = eu$, where $e \in E$ and u is a unit in $\mathbf{Z}[1/\lambda]$. Now f^k is a constant-to-one endomorphism, of degree $e^k u^k$. This forces some power of e, and therefore e itself, to be a unit in $\mathbf{Z}[1/\lambda]$. (See [**B1**, Corollary 2.4].) □

We do not know whether, in Theorems 1, 2 or the corollary, the condition that f be constant-to-one can be weakened to d-to-one almost everywhere. Nor do we know whether the condition on the multiplicity of λ is necessary in Theorems 1 or 2.

EXAMPLE. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then $\chi_A = (x-2)(x+1)^2$. Since the entropy of Σ_A is $\log 2$, Boyle's theorem tells us only that any endomorphism f of Σ_A must have degree a power of 2. If we assume that f is constant-to-one, of degree d, then since $\lambda = -1$ is an eigenvalue and the only units in $\mathbf{Z}[1/\lambda] = \mathbf{Z}$ are ± 1, by the corollary we must have $d = 1$; thus f must be an automorphism.

The rational units of $\mathbf{Z}[1/\lambda]$ can be computed from the following proposition.

PROPOSITION. *Let λ be an algebraic integer with minimal polynomial $r(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, and suppose $d \in \mathbf{Z}$. Then the following conditions are equivalent.*

(1) *d is a unit in $\mathbf{Z}[1/\lambda]$.*

(2) *If p is a rational prime dividing d, then p divides a_i, $0 \le i \le n-1$.*

(3) *d divides some power of the greatest common divisor of $\{a_i : 0 \le i \le n-1\}$.*

PROOF. See [**B1**, Proposition 2.3]. Now, by the corollary, if f is a constant-to-one endomorphism of Σ_A, of degree d, then d is a unit in $\mathbf{Z}[1/\lambda]$ for all nonzero eigenvalues λ. By the above proposition, this implies that if p is a rational prime dividing d, then p divides all the nonleading coefficients of the *characteristic* polynomial of A.

M. Boyle has pointed out that there is a partial converse to the corollary to Theorem 1.

THEOREM (BOYLE, [**B2**]). *Let A be a nonnegative, integral matrix, whose minimal polynomial has degree m, and suppose that d is a positive integer which is a unit in $\mathbf{Z}[1/\lambda]$, for every nonzero eigenvalue λ for A. If $n \geq m$, then there exists a constant-to-one endomorphism of Σ_{A^n} of degree d.*

PROOF. Suppose p is a prime dividing d. Let $q(x)$ denote the minimal polynomial for A. Since $q(x)$ divides χ_A, then by the preceding proposition, p divides every nonleading coefficient of $q(x)$. Since $q(x)$ is monic, of degree m, and $q(A) = 0$, it follows that p divides every entry of A^n, for all $n \geq m$. Let $B = (1/p)A^n$. Then $\Sigma_{A^n} \cong \Sigma_B \times \Sigma_{(p)}$, where $\Sigma_{(p)}$ is the full shift on p symbols. It is well known that $\Sigma_{(p)}$ admits a constant-to-one endomorphism of degree p. For example, let g be defined by the two block map $g(x_i x_{i+1}) = x_i + x_{i+1} \pmod p$, where the symbol set for $\Sigma_{(p)}$ is $\{0, 1, \ldots, p-1\}$. Let $\tilde{f}\colon \Sigma_B \times \Sigma_{(p)} \to \Sigma_B \times \Sigma_{(p)}$ be defined by $\tilde{f}(x, y) = (x, g(y))$. Then $\tilde{f}$ is constant-to-one, of degree p, and $\tilde{f}$ extends naturally to a constant-to-one endomorphism f_p of Σ_{A^n}, of degree p. Now the theorem follows by letting $f = f_{p_1}^{e_1} f_{p_2}^{e_2} \cdots f_{p_k}^{e_k}$ where $d = \prod_{i=1}^k p_{p_i}^{e_i}$ is the prime factorization of d. $\square$

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