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THE FRANKLIN SYSTEM AS SCHAUDER BASIS FOR $L^p_\mu[0,1]$

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ABSTRACT. The present work is devoted to a characterization of those spaces $L^p_\mu[0,1]$, $p \geq 1$, μ totally finite, for which $\mathcal{F}$, the Franklin system, is a Schauder basis. Because, in such cases, the measure μ must be absolutely continuous with respect to the Lebesgue measure, m, the necessary and sufficient condition is expressible in terms of a weight function, W, the Radon-Nikodym derivative of μ with respect to m. One finds that the spaces $L^p_\mu[0,1]$ for which $\mathcal{F}$ serves as Schauder basis are precisely those for which W satisfies the A_p condition introduced by Muckenhoupt. On the other hand, Krancberg has shown that a much less restrictive condition on μ is both necessary and sufficient for the Haar system, $\mathcal{H}$, to be a Schauder basis for $L^p_\mu[0,1]$. Thus, as a mildly surprising corollary of the theorem contained herein, one finds that the class of spaces $L^p_\mu[0,1]$, for which $\mathcal{H}$ is a Schauder basis, properly contains the corresponding class of spaces for which the Franklin system so serves.

1. One might suspect that the Haar and Franklin systems serve as Schauder bases for precisely the same Banach function spaces, since it is known that both do so serve in many of the classical spaces. This expectation is not fulfilled, however, for there prove to be standard Banach spaces for which the Haar system is a Schauder basis, but the Franklin system is not. These spaces belong to a family characterized by Krancberg [**6**] in the process of answering the following question posed by Olevskii: For a given p in $[1,+\infty)$ under what conditions on the totally-finite Borel measure μ will the Haar system constitute a Schauder basis for $L^p_\mu[0,1]$? Krancberg observed that μ necessarily would be absolutely continuous with respect to the Lebesgue measure, so that the question could be answered by specifying a set of restrictions on W, the corresponding Radon-Nikodym derivative of μ. A single condition was found to be both necessary and sufficient; viz.,

$$\sup_{\Delta} \frac{1}{|\Delta|} \left[\int_\Delta W(t)\,dt\right]^{1/p} \left[\int_\Delta [W(t)]^{-q/p}\,dt\right]^{1/q} = O(1),$$

in which the supremum is taken over the set of all dyadic intervals $\Delta = [(k-1)/2^m, k/2^m]$, $k = 1,\ldots,2^m$; $m = 0,1,\ldots$; and q is the conjugate of p.

Although the Franklin system is a Schauder basis for many of the spaces $L^p_\mu[0,1]$, the Krancberg condition is not sufficient to ensure this circumstance. Indeed, one finds that the Franklin system is a basis only for those spaces $L^p_\mu[0,1]$ for which the corresponding weight functions satisfy the condition A_p, introduced by Muckenhoupt [**7**].

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The present note is devoted to the proof of this proposition. As one would expect, the arguments are grounded in the seminal work of Ciesielski [**1**,**2**] on the Franklin system.

2. The Franklin system $\mathcal{F} = \{f_n\colon n = 0, 1, \dots\}$, herein considered, is obtained from the standard Schauder system via the Gram-Schmidt-orthonormalization process. Thus, the first two Franklin functions are given by the formulae

$$f_0(t) = 1, \qquad f_1(t) = \sqrt{3}(2t-1), \quad \text{for } t \in [0,1],$$

and, for $n = 2^m + r$, $m = 0, 1, \dots$; $r = 1, \dots, 2^m$, each f_n is both continuous on $[0,1]$ and linear on each dyadic interval $[t_{i-1}^{(n)}, t_i^{(n)}]$, where

$$t_i^{(n)} = \begin{cases} i/2^{m+1}, & i = 0, \dots, 2r; \\ (i-r)/2^m, & i = 2r+1, \dots, 2^m + r. \end{cases}$$

Moreover, the following fundamental results concerning the magnitudes of the individual Franklin functions and the associated Dirichlet kernels were established in [**1** and **2**].

THEOREM A (CIESIELSKI). *In the notation specified above, one has, for $n = 2^m + r$,*

$$c_1 2^{m/2} e^{-\alpha|i-(2r-1)|} < (-1)^{i+1} f_n(t_i^{(n)}) < c_2 2^{m/2} e^{-\alpha|i-(2r-1)|}$$

with $c_1 = (2+\sqrt{3})/3\sqrt{3}$, $c_2 = 4\sqrt{3}(2+\sqrt{3})$ and $\alpha = \ln(2+\sqrt{3})$. Moreover,

$$\|f_n\|_\infty = \begin{cases} |f_n(0)|, & \text{if } r = 1; \\ f_n(t_{2r-1}^{(n)}), & \text{if } 1 < r < 2^m; \\ |f_n(1)|, & \text{if } r = 2^m. \end{cases}$$

THEOREM B (CIESIELSKI). *Let K_n denote the nth Franklin-Dirichlet kernel. Then, for $n \geq 2$ and $0 \leq s,\ t \leq 1$,*

$$|K_n(s,t)| \leq bne^{-\alpha n|t-s|/2},$$

with $b = 25\sqrt{3}/2$.

Because the set of all finite linear combinations of Franklin functions is dense in $C[0,1]$ (indeed, $\mathcal{F}$ is a Schauder basis for $C[0,1]$), the following proposition shows that the first step in the Krancberg program is also an appropriate starting point for the present discourse.

LEMMA C (KRANCBERG). *Let μ be a Borel measure on $[0,1]$. If $\{\varphi_n\colon n = 1, 2, \dots\}$ is an orthonormal system that is both closed in $C[0,1]$ and a Schauder basis for some $L^p_\mu[0,1]$, $p \geq 1$, then μ is absolutely continuous with respect to the Lebesgue measure.*

For the remainder of the present work, however, one is obliged to deal with a proper subset of the class of $L^p_\mu[0,1]$-spaces considered by Krancberg.

DEFINITION D (MUCKENHOUPT). The nonnegative, Lebesgue-integrable function W satisfies the condition A_p, $p \geq 1$, with constant C iff $0 < W(t) < +\infty$ on a set of full measure, $C < +\infty$, and, for every interval $I \subset [0,1]$,

$$\frac{1}{|I|}\left(\int_I W(t)\,dt\right)^{1/p}\left(\int_I [W(t)]^{-q/p}\,dt\right)^{1/q} \leq C.$$

Here q is the conjugate of p, and, for $p = 1$, the second factor is understood to be $\operatorname{ess\,sup}\{[W(t)]^{-1}\colon\ t \in I\}$.

It follows directly from the definition that if W satisfies A_p for some $p > 1$, then $W^{-q/p}$ satisfies A_q with the same constant. Moreover, the following critical result can be found in [**4**].

LEMMA E (HUNT, MUCKENHOUPT, WHEEDEN). *If, for some p in $(1, +\infty)$, W satisfies A_p with constant C, then there exists a constant K, depending only upon p and C, such that, for every subinterval I of $[0,1]$,*

$$|I|^p \int_{[0,1]\setminus I} \frac{W(t)}{|t - a_I|^p}\, dt \le K \int_I W(t)\, dt,$$

where a_I denotes the center of I. If W satisfies A_1, with constant C, then there exists a constant K, depending only upon C, such that, for every subinterval I of $[0,1]$,

$$|I|^2 \int_{[0,1]\setminus I} \frac{W(t)}{|t - a_I|^2}\, dt \le K \int_I W(t)\, dt.$$

3. Let μ be a totally-finite Borel measure on $[0,1]$. By virtue of Lemma C, the Franklin system can serve as Schauder basis for some space $L^p_\mu[0,1]$, $p \ge 1$, only if μ is absolutely continuous with respect to the Lebesgue measure; thus, for the present discussion, one may assume that there is a nonnegative, Lebesgue-measurable function W such that, for each Lebesgue-measurable set E,

$$\mu(E) = \int_E W(t)\, dt.$$

THEOREM. *The Franklin system is a Schauder basis for $L^p_\mu[0,1]$, for some p in $[1, +\infty)$, if and only if W satisfies A_p.*

The proof, for p in $(1, +\infty)$, follows. The proof for the remaining case is somewhat simpler and has been omitted.

The condition is sufficient. A portion of this conclusion could be deduced from a known, but apparently unpublished, result; viz., if $p > 1$ and if W satisfies the A_p condition, then the Franklin system is a Schauder basis for the weighted Hardy space $H^p(W)$. The desideratum would follow, in these cases, from the equivalence of the corresponding Hardy and Lebesgue spaces [**3**]. Under the circumstances, it seems to be appropriate to include the direct argument given below.

Since the set of all finite linear combinations of Franklin functions is dense in each space $L^p_\mu[0,1]$, it suffices to show that the Fourier-Franklin, partial-sum transformations are uniformly-bounded operators on $L^p_\mu[0,1]$. Let f be an element of $L^p_\mu[0,1]$. From Hölder's inequality,

$$\int_0^1 |f(t)|\, dt \le \left(\int_0^1 |f(t)|^p W(t)\, dt \right)^{1/p} \left(\int_0^1 [W(t)]^{-q/p}\, dt \right)^{1/q},$$

one sees that $L^p_\mu[0,1]$ is a subset of $L^1[0,1]$ so that

$$S_n f = \int_0^1 K_n(\cdot, t) f(t)\, dt$$

is defined for each f in $L^p_\mu[0,1]$. Now let

$$\Delta_i = [t^{(n)}_{i-1}, t^{(n)}_i], \qquad i = 1, \ldots, n;$$

let

$$k_{ji} = \sup\{|K_n(x,t)|\colon\ (x,t) \in \Delta_j \times \Delta_i\}, \qquad i,j = 1, \ldots, n;$$

and let

$$\varphi_j = \frac{1}{n}\sum_{i=1}^{n} k_{ji}\chi_{\Delta_i}, \qquad j = 1, \ldots, n,$$

where χ_E denotes the characteristic function of E. From Theorem B, one deduces that there is a constant D, independent of n, such that $\sum_{j=1}^{n} \varphi_j(t) \le D$, for all t in $[0,1]$. One has, for f in L^p_μ,

$$\begin{aligned}
&\int_0^1 \left| \int_0^1 K_n(x,t) f(t)\, dt \right|^p d\mu(x) \\
&\quad \le \int_0^1 \left\{ \int_0^1 |K_n(x,t)|\,|f(t)|\, dt \right\}^p d\mu(x) \\
&\quad = \int_0^1 \left\{ \sum_{i=1}^{n} \int_{\Delta_i} |K_n(x,t)|\,|f(t)|\, dt \right\}^p d\mu(x) \\
&\quad \le \int_0^1 \left\{ \sum_{i=1}^{n} \int_{\Delta_i} |f(t)| \sum_{j=1}^{n} k_{ji}\chi_{\Delta_j}(x)\, dt \right\}^p d\mu(x) \\
&\quad = \int_0^1 \left\{ \sum_{j=1}^{n} \left[\sum_{i=1}^{n} k_{ji} \int_{\Delta_i} |f(t)|\, dt \right] \chi_{\Delta_j}(x) \right\}^p d\mu(x) \\
&\quad = \int_0^1 \sum_{j=1}^{n} \left\{ \int_0^1 |f(t)| \sum_{i=1}^{n} k_{ji}\chi_{\Delta_i}(t)\, dt \right\}^p \chi_{\Delta_j}(x)\, d\mu(x) \\
&\quad = n^p \int_0^1 \sum_{j=1}^{n} \left\{ \int_0^1 |f(t)|\varphi_j(t)\, dt \right\}^p \chi_{\Delta_j}(x)\, d\mu(x) \\
&\quad = n^p \sum_{j=1}^{n} \left\{ \int_0^1 |f(t)|\varphi_j(t)\, dt \right\}^p \mu(\Delta_j) \\
&\quad \le n^p \sum_{j=1}^{n} \left\{ \int_0^1 |f(t)|[\varphi_j(t)W(t)]^{1/p}[\varphi_j(t)]^{1/q}[W(t)]^{-1/p}\, dt \right\}^p \mu(\Delta_j) \\
&\quad \le n^p \sum_{j=1}^{n} \int_0^1 |f(t)|^p \varphi_j(t)\, d\mu(t) \left\{ \int_0^1 \varphi_j(t)[W(t)]^{-q/p}\, dt \right\}^{p/q} \mu(\Delta_j).
\end{aligned}$$

By virtue of Theorem B, there exists a constant B such that, for each j,

$$\varphi_j(t) \le \begin{cases} B|\Delta_j|^q/|t - a_{\Delta_j}|^q, & \text{if } t \in [0,1]\backslash\Delta_j; \\ B, & \text{if } t \in \Delta_j; \end{cases}$$

thus, from Lemma E, applied to $W^{-q/p}$,

$$\int_0^1 \varphi_j(t)[W(t)]^{-q/p}\,dt \le B(K+1)\int_{\Delta_j}[W(t)]^{-q/p}\,dt,$$

for each j. Moreover,

$$\left[\int_{\Delta_j}[W(t)]^{-q/p}\,dt\right]^{p/q}\mu(\Delta_j) = \left[\left(\int_{\Delta_j}W(t)\,dt\right)^{1/p}\left(\int_{\Delta_j}[W(t)]^{-q/p}\,dt\right)^{1/q}\right]^p$$

$$\le C^p|\Delta_j|^p \le 2^pC^pn^{-p},$$

where C is the constant associated with the condition A_p. Thus, with

$$A = 2C[B(K+1)]^{1/q}D^{1/p},$$

$$||S_nf||^p_{L^p_\mu} \le \frac{A^p}{D}\sum_{j=1}^n\int_0^1 |f|^p\varphi_j\,d\mu$$

$$= \frac{A^p}{D}\int_0^1 |f|^p\sum_{j=1}^n\varphi_j\,d\mu \le A^p||f||^p_{L^p_\mu}.$$

The condition is necessary. Again, following the route marked by Krancberg, if $\mathcal{F}$ is a Schauder basis for $L^p_\mu[0,1]$, then there exists a (conjugate) system $\{\theta_n\colon n = 0,1,\dots\}$, contained in $L^q_\mu[0,1]$, such that

$$\int_0^1 f_n\theta_m\,d\mu = \delta_{nm}, \qquad n,m = 0,1,\dots.$$

(See, for example [5].) Thus, for each f in $L^p_\mu[0,1]$, one has

$$f = \sum_{k=0}^\infty \theta_k(f)f_k.$$

For each $n = 0,1,\dots$, let $T_n\colon L^p_\mu[0,1] \to L^p_\mu[0,1]$ be defined by

$$T_nf = \sum_{k=0}^n \theta_k(f)f_k, \quad \text{for all } f \in L^p_\mu[0,1].$$

By virtue of the uniform-boundedness principle, $\{||T_n||\}_{n=0}^\infty$ is bounded, and, thus also, there exists a constant C such that

$$||f_n||_{L^p_\mu}||\theta_n||_{L^q_\mu} = ||T_n - T_{n-1}|| < C, \qquad \text{for all } n = 1,2,\dots.$$

From the completeness of $\mathcal{F}$ in $L^1[0,1]$ it follows that

$$\theta_n = f_nW^{-1}, \qquad \text{for all } n.$$

Thus,

$$||f_n||_{L^p_\mu}||\theta_n||_{L^q_\mu} = \left(\int_0^1 |f_n(t)|^pW(t)\,dt\right)^{1/p}\left(\int_0^1 |f_n(t)|^q[W(t)]^{-q/p}\,dt\right)^{1/q},$$

for $n = 1,2,\dots$. Therefore, in order to complete the proof, one only need demonstrate the existence of a positive constant K such that, for each subinterval I of $[0,1]$,

$$\inf\{|f_n(t)|\colon t \in I\} \ge K|I|^{-1/2}, \quad \text{for at least one } n.$$

The search for such a K eventually leads one to the following proposition.

LEMMA. *Let $H = [(r-1)2^{-m}, r2^{-m}]$, for some m and r satisfying $m = 2, 3, \dots$; $r = 1, \dots, 2^m$; and let $f_H = f_{2^m+r}$ be the corresponding element of the mth block of the Franklin system. Let a and b be the zeros of f_H on $[(r-1)2^{-m}, (2r-1)2^{-m-1}]$ and $[(2r-1)2^{-m-1}, r2^{-m}]$, and, if $r \neq 2^m$, let c be the zero of f_H on $[r2^{-m}, (r+1)2^{-m}]$. Let y_1, x_1, x_2, y_2 be the lengths of the following four subintervals of H: $[(r-1)2^{-m}, a]$, $[a, (2r-1)2^{-m-1}]$, $[(2r-1)2^{-m-1}, b]$, $[b, r2^{-m}]$, and, in the applicable cases, let y_3 be the length of $[r2^{-m}, c]$. The following relations obtain*

$$y_i \geq \left(\frac{c_1}{c_1 + c_2 e^{\alpha}}\right)\left(\frac{1}{2}|H|\right) \geq \frac{|H|/2}{136}, \qquad i = 1, 2; \tag{1}$$

$$\text{if } H \subset (0,1), \text{ then } x_i \geq \tfrac{1}{4}|H|, \qquad i = 1, 2; \tag{2}$$

$$\text{if } r = 1, \text{ then } y_1 \geq \tfrac{1}{4}|H|; \text{ if } r = 2^m, \text{ then } y_2 \geq \tfrac{1}{4}|H|; \tag{3}$$

$$\text{if } r \neq 2^m, \text{ then } y_3 \geq \left(\frac{c_1}{c_1 + c_2 e^{-\alpha}}\right)|H| > \frac{1}{12}|H|. \tag{4}$$

PROOF OF THE LEMMA. Since f_H is piecewise linear, each of the estimates follows from an application of Theorem A. For example,

$$\frac{y_2}{2^{-m-1}} = \frac{|f_H(r2^{-m})|}{|f_H(r2^{-m})| + f_H((2r-1)2^{-m-1})} \geq \frac{c_1 e^{-\alpha}}{c_1 e^{-\alpha} + c_2},$$

where the proportion is deduced from the observation of two similar triangles. The estimates for the other y_i have similar derivations. The inequalities for the x_i follow even more swiftly, for, if $H \subset (0,1)$, then f_H assumes its maximum value at $(2r-1)2^{-m-1}$.

Since, for each interval I with $|I| \geq \varepsilon > 0$,

$$\begin{aligned}\frac{1}{|I|}&\left(\int_I W(t)\,dt\right)^{1/p}\left(\int_I [W(t)]^{-q/p}\,dt\right)^{1/q} \\ &\leq \frac{1}{\varepsilon}\left(\int_0^1 W(t)\,dt\right)^{1/p}\left(\int_0^1 [W(t)]^{-q/p}\,dt\right)^{1/q},\end{aligned}$$

it follows that one need establish the A_p-inequality only for those subintervals whose lengths are smaller than some preassigned positive number ε_0. With this fact in mind, one takes $\varepsilon_0 = 2^{-24}$.

Thus, it will suffice to establish the existence of a positive constant B such that, for every interval I satisfying $2^{-m-1} \leq |I| < 2^{-m}$, with $m \geq 24$, there exist natural numbers l and s, with $l \geq m - 24$ and $1 \leq s \leq 2^l$, such that

$$|f_{2^l+s}(t)| \geq B2^{l/2}, \quad \text{for all } t \text{ in } I.$$

For, if I, l and s are as above, and if $n = 2^l + s$, then

$$\begin{aligned}||f_n||_{L^p_\mu}||\theta_n||_{L^q_\mu} &= \left(\int_0^1 |f_n(t)|^p W(t)\,dt\right)^{1/p}\left(\int_0^1 |f_n(t)|^q [W(t)]^{-q/p}\,dt\right)^{1/q} \\ &\geq B^2 2^l \left(\int_I W(t)\,dt\right)^{1/p}\left(\int_I [W(t)]^{-q/p}\,dt\right)^{1/q},\end{aligned}$$

from which it follows that

$$\frac{1}{|I|}\left(\int_I W(t)\,dt\right)^{1/p}\left(\int_I [W(t)]^{-q/p}\,dt\right)^{1/q} \le K,$$

with $K = 2^{25}CB^{-2}$.

Hence, let I be a subinterval of $[0,1]$ such that $2^{-m-1} \le |I| < 2^{-m}$, with $m \ge 24$. One distinguishes two cases. In the first of these, one supposes that I is a subset of a dyadic interval, J, of length 2^{-m}, and one considers the progenitors of J in various dyadic partitions of $[0,1]$. Let $\pi_k = \{[(i-1)/2^k, i/2^k]\colon\ i = 1,\dots,2^k\}$, for $k = 0,1,\dots$. Then J belongs to π_m, and there is an unique sequence $\{J_{m-k}\}_{k=0}^{m}$ such that J_{m-k} belongs to π_{m-k}, for each k, $J = J_m \subset J_{m-1} \subset \cdots \subset J_0 = [0,1]$, and, for $k < m$, $J_{m-k-1} = J_{m-k} \cup J^*_{m-k}$, where J^*_{m-k} is an element of π_{m-k} that abuts J_{m-k} either on the right or on the left. It is convenient to indicate these single-generational changes by R and L, and to indicate a transition to a generation several steps removed from that of J by a word $w = w(J)$ of the form $XX\cdots X$, in which each X is either an R or an L.

Suppose now that, in the twelve generations that precede J's generation, no ancestor of J contains either 0 or 1.

Suppose further that J's ancestral sequence is represented by the word $w = LL\cdots LR\cdots$ in which k, the initial number of L's in w, satisfies $2 \le k \le 8$. The dyadic interval in π_{m-k-1} to which this word corresponds is $J_{m-k-1} = [\tau - 2^k|J|, \tau + 2^k|J|]$, where τ is the right endpoint of J. Since, by assumption, this interval lies within the interior of $[0,1]$, there is a Franklin function $f_{J,k+1}$, in the $(m-k-1)$st block of the Franklin system, such that $|f_{J,k+1}|$ achieves its greatest value at τ. Hence, it follows, from Theorem A (and the Lemma), that

$$|f_{J,k+1}(t)| \ge \tfrac{1}{2}c_1 2^{(m-(k+1))/2},$$

for all t in J and, a fortiori, for all t in I.

On the other hand, if, in $w(J)$, nine consecutive, initial L's appear, one considers f_{J9}, the element of the $(m-9)$th Franklin block that corresponds to the dyadic interval $J_{m-9} = [\tau - 2^9|J|, \tau]$ (as indicated in the Lemma). According to the Lemma, the zero of f_{J9} that lies in the right half of J_{m-9} is at least $\frac{3}{2}|J|$ units to the left of τ. It follows from Theorem A that

$$f_{J9}(t) \le -\tfrac{1}{3}(c_1 e^{-\alpha})2^{(m-9)/2}, \quad \text{for all } t \text{ in } J;$$

thus, a fortiori,

$$|f_{J9}(t)| \ge \tfrac{1}{3}(c_1 e^{-\alpha})2^{(m-9)/2}, \quad \text{for all } t \text{ in } I.$$

Hence, in either case, a word beginning with two L's leads to a Franklin function, f, such that $l(f)$, the rank of f, is not less than $m-9$, and

$$|f(t)| \ge B2^{l(f)/2}, \quad \text{for all } t \text{ in } I,$$

with $B = \frac{1}{3}c_1 e^{-\alpha}$.

A symmetrical argument shows that a word of the form $w = RR\cdots$ can be associated with a Franklin function of rank $m-9$ or greater, that satisfies an inequality of the same type.

Next, one partitions the set of all words beginning with LR into four classes, the membership of a word in a given class being determined by the first five (or fewer) letters of the word. These classes together with their initial sequences of letters are (Γ_1, LRR), $(\Gamma_2, LRLL)$, $(\Gamma_3, LRLRR)$, $(\Gamma_4, LRLRL)$. By means of arguments very nearly identical to that given above, one deduces the existence of natural numbers, k_1, k_2, k_3, k_4, such that no k_i exceeds 12 and such that, for each i, if w belongs to Γ_i, then there exists a Franklin function f_{Jk_i} for which $l_i = l(f_{Jk_i}) \geq m - k_i$, and

$$|f_{Jk_i}(t)| \geq B2^{l_i/2}, \quad \text{for all } t \text{ in } I.$$

Since the process, in which one is engaged here, is more akin to accounting than it is to mathematics, it seems to be appropriate to omit the details.

By virtue of the symmetry of the situation, if $w(J)$ begins with RL, one obtains the same estimates without making a further investment of labor.

Of course it may happen that J's ancestral sequence leads to the boundary of $[0,1]$ in twelve generations or fewer. In this case one can show that there is a Franklin function of rank not less than $m-15$ that satisfies an inequality of the required type. For example, suppose that, for some $k \leq 12$, 1 belongs to J_{m-k}, the ancestor of J in π_{m-k}. Then, $J_{m-k} = [1-2^{k-m}, 1]$, $J_{m-k-3} = [1-2^{k+3-m}, 1]$, and, by virtue of assertion (3) of the Lemma, it follows that

$$|f_{2^{m-k-2}}(t)| \geq B2^{(m-k-3)/2}, \quad \text{for all } t \text{ in } I.$$

All of the preceding is predicated upon the assumption that I should be a subset of a dyadic interval of length not greater than $2|I|$, a circumstance that is not certain to exist. Thus, suppose, finally, that $2^{-m-1} \leq |I| < 2^{-m}$, but that I is not a subset of any element of π_m. (Consider, for example, $I = [\frac{1}{2} - 2^{-m-2}, \frac{1}{2} + 2^{-m-2}]$.) In this case, one considers the dyadic ancestors of $J = [\sigma, \tau]$, the element of π_m that contains the left endpoint of I. If $w(J)$ begins with a sequence of eight or fewer L's followed by an R, then I is a subset of a dyadic interval J_{m-k} in π_{m-k}, with $1 \leq k \leq 9$. Thus, the procedure developed above can be employed to demonstrate the existence of a Franklin function f such that $l(f) \geq m-24$, and

$$|f(t)| \geq B2^{l(f)/2}, \quad \text{for all } t \text{ in } I.$$

If $w(J)$ begins with a sequence of nine L's, one uses the first and fourth assertions of the Lemma to show that $f = f_{J9}$ satisfies $f(\sigma) < -\frac{1}{3}f(\tau)$ and $f(\tau + |J|) < -\frac{1}{2}f(\tau)$, from which it follows that

$$|f(t)| \geq B2^{l(f)/2}, \quad \text{for all } t \text{ in } I.$$

4. The following elementary example gives an indication of the width of the gap between the Krancberg hypothesis and the A_p condition.

Let $W\colon [0,1] \to \mathbf{R}$ be defined as

$$W(t) = \begin{cases} (1-2t)^{-1/2}, & \text{if } 0 \leq t < \frac{1}{2}, \\ (2t-1)^{-1/2}, & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

Relatively simple calculations suffice to show that W does not satisfy A_2, while the corresponding Krancberg condition is trivially fulfilled.

COROLLARY. *The class of spaces $L^p_\mu[0,1]$ for which the Haar system serves as Schauder basis, properly contains the corresponding class of spaces associated with the Franklin system.*

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