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FINITE DIMENSIONAL COMPLEMENT THEOREMS: EXAMPLES AND RESULTS

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ABSTRACT. Examples are given which show the necessity of various hypotheses in the known finite dimensional complement theorems. In addition, several positive results are presented which improve one direction of such theorems.

0. Introduction. Since Chapman [**Ch**$_1$] proved that Z-sets in Q have the same shape if and only if they have homeomorphic complements, there has been a great deal of interest in establishing related finite dimensional "complement theorems." For a survey of results in the area, see [**Sh**$_1$ and **Sh**$_2$].

In this paper, we offer some examples and results involving complement theorems in the euclidean sphere S^n. Specifically, in §§1–3 we give the following:

(1) Examples of r-connected polyhedra $X, Y \subset S^n$ of fundamental dimension k, where $6 \leq n = 2k - r$ and $k \leq n - 3$, such that $X \cong Y$ but $S^n - X \not\cong S^n - Y$.

(2) Examples of polyhedra $X \subset S^n, n \geq 5$, of fundamental dimension $[\frac{n}{2}] + 1$ such that $X \not\simeq S^1$ but $S^n - X \cong \mathrm{Int}(S^{n-2} \times D^2) \cong S^n - S^1$.

(3) Improved conditions under which $S^n - X \cong S^n - Y$ implies $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

The examples of §1 show that the connectivity conditions stated in Theorem C of [**ISV**], Theorem 1 of [**HI**$_2$], and Theorem 5.1 of [**St**$_1$] are necessary. The examples of §2 derive from an explicit computation of the fundamental dimension of knot exteriors. Other examples of pairs of continua having different shapes but homeomorphic complements are known, but none have been observed with such large codimension; in fact, all such examples that the authors have seen involve at least one continuum whose fundamental dimension is at least $n-2$. In §3 we obtain some positive results, including an improvement on Theorem 4 of [**ISV**]; this result allows us to replace a codimension 4 condition in the main results of [**ISV**] by the more natural codimension 3 condition.

The definitions and notations used herein will be familiar to workers in the field. They may be found in [**ISV**, **Sh**$_1$, or **Ve**$_2$]. We use "$\cong$" to denote either "is homeomorphic to" or "is isomorphic to" (depending on the context) and "$\simeq$" to denote either "is homotopic to" or "is homotopy equivalent to" (depending on the context).

1. Continua in S^n with the same shape but different complements. Let $A \subset S^{p+q+1}$ denote the union of two disjoint unknotted and unlinked PL spheres

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of dimensions p and q, so that $S^{p+q+1} - A \simeq S^{p+q} \vee S^p \vee S^q$. Let $B \subset S^{p+q+1}$ denote the union of two disjoint unknotted and linked PL spheres of dimensions p and q, so that $S^{p+q+1} - B \simeq S^p \times S^q$. For $i = 1, 2, \ldots$, let $X = \Sigma^i A \subset \Sigma^i S^{p+q+1} = S^{p+q+i+1}$ and $Y = \Sigma^i B \subset S^{p+q+i+1}$, where Σ denotes suspension. Then $X \cong Y \simeq S^{p+i} \vee S^{q+i} \vee S^i$. We may assume $p \leq q$, so X and Y are $(i-1)$-connected and have fundamental dimension $q+i$. It is easy to see that $S^{p+q+i+1} - X \simeq S^{p+q} \vee S^p \vee S^q \not\simeq S^p \times S^q \simeq S^{p+q+i+1} - Y$. Letting $p = q \geq 2$, we thus have the following.

(1.1) THEOREM. *For $q \geq 2$, $i \geq 1$, and $n = 2q + i + 1$, there exist $(i-1)$-connected polyhedra $X, Y \subset S^n$ of fundamental dimension $q + i \leq n - 3$ such that $X \cong Y$ but $S^n - X \not\cong S^n - Y$.*

In order to see how Theorem 1.1 relates to the results of [**ISV**], suppose that X and Y are subpolyhedra of S^n, $n \geq 5$, such that $\mathrm{Fd}(X) = k \leq n - 3$ and $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. The question is

(1.2) *Is it the case that $S^n - X \cong S^n - Y$?*

Case 1. $2k + 1 > n$. In Theorem 1.1, let $q = n - k - 1$ and let $i = 2k - n + 1$. We see that for each k in this range we have k-dimensional examples for which the complements are not homeomorphic. The examples are $(i-1) = (2k-n)$-connected. Were such continua to be $(2k - n + 1)$-connected, then by Theorem C of [**ISV**] their complements would be homeomorphic.

Case 2. $2k + 1 = n$. The remark at the top of p. 214 of [**ISV**] shows that there exist X and Y with different complements. However, if X and Y are connected, then Theorem C if [**ISV**] shows that $S^n - X \cong S^n - Y$.

Case 3. $2k + 1 < n$. The trivial range theorem of [**Ve$_1$**] shows that $S^n - X \cong S^n - Y$.

(1.3) REMARK. These examples also show the necessity of the connectivity conditions in Theorem 1 of [**Hl$_2$**] and Theorem 5.1 of [**St$_1$**], for if X and Y in Theorem 1.1 were shape concordant, then their complements would be homeomorphic by the complement theorem of [**Sh$_3$**] (or that of [**LV**]).

The authors wish to thank Steve Ferry for enlightening conversations regarding the examples described above.

2. Continua in S^n with different shapes but homeomorphic complements. If Σ is a locally flat PL n-sphere in S^{n+2}, by an *exterior* of Σ we mean a subpolyhedron of S^{n+2} whose complement is the interior of a regular neighborhood of Σ. Since any two exteriors are PL homeomorphic, we may simply speak in what follows of *the* exterior of Σ, which we shall denote by $\mathrm{Ext}(\Sigma)$.

(2.1) THEOREM. *Suppose Σ is a locally flat PL n-sphere in S^{n+2}. Let*

$$k = \begin{cases} \text{l.u.b.}\{i \mid \pi_j(S^{n+2} - \Sigma) \cong \pi_j(S^1) \text{ for } 0 \leq j \leq i\} & \text{if such a l.u.b. exists,} \\ n & \text{otherwise.} \end{cases}$$

Then

(a) $0 \leq k < \frac{1}{2}(n+1)$ *or* $k = n$, *and*

(b) $\mathrm{Fd}(\mathrm{Ext}(\Sigma)) = n - k + 1$.

PROOF. Let $U = \mathrm{Ext}(\Sigma)$ and let K be a subpolyhedron of U such that $U \searrow K \subset \mathrm{Int}(U)$. Then $\pi_i(U) \cong \pi_i(S^{n+2} - \Sigma)$ and $U - K \cong \partial U \times [0,1) \simeq \partial U \cong S^1 \times S^n$, so $\pi_1(U - K) \cong \mathbf{Z}$ and $\pi_i(U-K) = 0$ if $2 \le i \le n-1$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(U-K) & \xrightarrow{\alpha} & \pi_1(U) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\mu} \\ H_1(U-K) & \xrightarrow{\beta} & H_1(U) \end{array}$$

where α, β are induced by the inclusions, and η, μ are the Hurewicz homomorphisms. Then η and β are isomorphisms. Thus $\mu\alpha$ is an isomorphism, and so α is 1-1. We note that this latter fact implies that K satisfies the inessential loops condition (ILC) in S^{n+2}. (See [**Ve**$_1$] for the definition of ILC.) We also observe that α is onto if and only if $k \ge 1$.

Notice that the theorem is trivially true if $n = 0$, for in this case U is an annulus and $k = 0$. When $n = 1$, the conclusion of the theorem follows easily from the well-known consequence of Dehn's Lemma which states that Σ is unknotted if and only if $\pi_1(U) \cong \mathbf{Z}$. When $n = 2$ and $\pi_1(U) \cong \mathbf{Z}$, then $U \cong S^1 \times D^3$ by Theorem 6 of [**Fr**], and the result follows. When $n = 2$ and $\pi_1(U) \not\cong \mathbf{Z}$, then U is not aspherical by [**DV**], and so $2 \le \mathrm{Fd}(U) \le 3$. If $\mathrm{Fd}(U) = 2$, then $\pi_1(U, U-K) = 0$ by [**Ve**$_2$], and so it follows from the exact homotopy sequence of the pair $(U, U-K)$ that α is onto; by the observation at the end of the preceding paragraph this shows that $k \ge 1$, contradicting the assumption that $\pi_1(U) \not\cong \mathbf{Z}$. Therefore $\mathrm{Fd}(U) = 3$ and the conclusion of the theorem holds. We have thus established the validity of the theorem for $0 \le n \le 2$, and may assume in the remainder of the proof that $n \ge 3$.

Now suppose $0 \le k < \frac{1}{2}(n+1)$. It follows from the exact homotopy sequence of the pair $(U, U-K)$, the definition of k, and the properties of α established in the first paragraph that $\pi_{k+1}(U, U-K) \ne 0$ and $\pi_i(U, U-K) = 0$ if $0 \le i \le k$. Since $k \le n-1$ and $n \ge 3$, we may apply Stallings' Engulfing Theorem [**St**$_2$] to engulf the k-skeleton of U by $U - K$. Consequently, K may be engulfed by a regular neighborhood of the dual $(n-k+1)$-skeleton of U. It follows from this that $\mathrm{Fd}(U) \le n-k+1$. If $\mathrm{Fd}(U) \le n-k$, then by [**Ve**$_2$] and the above-mentioned fact that K satisfies ILC in U, K has arbitrarily small PL manifold neighborhoods with spines of dimension at most $n-k$. But polyhedra of dimension $k+1$ may be moved off such spines by general position, thereby yielding $\pi_{k+1}(U, U-K) = 0$, a contradiction. Thus $\mathrm{Fd}(U) = n-k+1$.

It is a well-known result of Levine [**Le**] that if $\pi_i(S^{n+2} - \Sigma) \cong \pi_i(S^1)$ for $i \le \frac{1}{2}(n+1)$, then Σ is unknotted. Thus the case $\frac{1}{2}(n+1) \le k < n$ cannot occur, and $\mathrm{Fd}(U) = 1$ when $k = n$. This establishes the theorem; we show in the next paragraph, however, how to complete the proof without appealing to Levine's Theorem.

First note that the proof given for the case $0 \le k < \frac{1}{2}(n+1)$ actually shows that $\mathrm{Fd}(U) = n-k+1$ whenever $k \le n-2$. We show the following:

(∗) *If there exists a j such that $\pi_i(U) \cong \pi_i(S^1)$ for $i \le j$ but $\pi_{j+1}(U) \not\cong \pi_{j+1}(S^1)$, then $j \le n-2$.*

Suppose (∗) is not true, i.e., that $j \ge n-1$. Then the pair $(U, U-K)$ is $(n-1)$-connected, and so the argument in the third paragraph of the proof shows that

$\mathrm{Fd}(U) \le 2$. We let $S^1 \subset K$ be an embedded generator of $H_1(U)$. Then $\pi_i(U, S^1) = 0$ for $i \le 2$, so we can use Stalling's Engulfing Theorem (we are still assuming $n \ge 3$) to find a regular neighborhood N of S^1 such that $S^1 \subset K \subset N \subset U$. It follows that $K \simeq S^1$, contradicting the supposition that $\pi_{j+1}(U) \not\cong \pi_{j+1}(S^1)$. Thus $(*)$ is true and we see that either $k \le n-2$ or $U \simeq S^1$. This establishes part (b) of the conclusion of the theorem. To establish part (a), suppose $k \ge \frac{1}{2}(n+1)$ (equivalently, $k \ge n-k+1 = \mathrm{Fd}(U)$) and let $S^1 \subset K$ be an embedded generator of $H_1(U)$ as above. Then $\pi_i(U, S^1) = 0$ for $i \le n-k+1$, and we again conclude that $K \simeq S^1$ and $k = n$. $\square$

(2.2) THEOREM. *There exists a polyhedron* $X \subset S^n$, $n \ge 5$, *such that* $\mathrm{Fd}(X) = [\frac{n}{2}] + 1$, $X \not\simeq S^1$ *and* $S^n - X \cong \mathrm{Int}(S^{n-2} \times D^2) \cong S^n - S^1$.

PROOF. The examples are knot exteriors. They come about by applying Theorem 2.1 to the examples in [**Su**] of

(a) a locally flat PL $(2q-1)$-sphere in S^{2q+1} with $q \ge 2$ and $k = q-1$, and
(b) a locally flat PL $(2q)$-sphere in S^{2q+2} with $q \ge 1$ and $k = q-1$. $\square$

(2.3) REMARK. In a result such as Theorem 2.2, it is desirable to have the fundamental dimensions be as small as possible. By the trivial range theorem of [**Ve**$_1$], it is not possible to replace the condition $\mathrm{Fd}(X) = [\frac{n}{2}] + 1$ in Theorem 2.2 by $\mathrm{Fd}(X) = [\frac{n}{2}] - 1$. We also observe that we cannot reduce $\mathrm{Fd}(X)$ to $[\frac{n}{2}]$ by using knot exteriors, because to achieve $\mathrm{Fd}(X) = j$ when $n = 2j+1$, we must have $k = (2j-1) - j + 1 = j \ge \frac{1}{2}(2j-1+1)$, and this is impossible by conclusion (a) of Theorem 2.1. Likewise, to achieve $\mathrm{Fd}(X) = j$ when $n = 2j$, we must have $k = (2j-2) - j + 1 = j - 1$, so $\pi_i(X) \cong \pi_i(S^1)$ for $0 \le i \le j-1$. But then $\pi_i(X) \cong \pi_i(S^1)$ for $i \le \frac{1}{2}(2j-1)$, and so $X \cong S^1 \times D^{n-1}$ by Levine's Theorem. In the next section we prove a result that gives further restrictions on the possibility of lowering the fundamental dimension for continua in S^n.

3. Conditions under which $S^n - X \cong S^n - Y$ **implies** $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. Let X and Y be compacta in S^n. The examples in §1 show that it is not necessarily the case that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ implies $S^n - X \cong S^n - Y$. When we combine the examples with the theorems in [**ISV**], we get a fairly complete picture of what can happen, at least for compacta of polyhedral shape, in the sense that all possible fundamental dimensions and connectivities are covered by either the theorems or the examples. The only remaining unanswered questions have to do with compacta which do not have polyhedral shape; see [**Sh**$_2$, Question 2].

The examples in §2, however, do not fit as well with the converses of the theorems. First, all the examples are knot complements and have $\pi_1 \cong \mathbf{Z}$; we do not know of any simply connected examples. Second, fundamental dimension $[\frac{n}{2}]$ is not covered by either the theorems or the examples; cf. Remark 2.3 above. This leads us to suspect that the theorems are too weak and can be improved. In this section we present the results of our efforts to fill in some of these gaps.

Our first result is an improved version of [**ISV**, Theorem 4]. It improves that theorem by eliminating both the codimension and the ILC hypotheses. We still have two hypotheses: we assume that the compacta are 1-shape connected and that they lie in a hyperplane. As far as we know, either one of these hypotheses by itself might suffice; see [**Sh**$_2$, Question 1].

(3.1) THEOREM. *Let X and Y be 1-shape connected compacta in $E^n = E^n \times \{0\} \subset E^{n+1}$. Then $E^{n+1} - X \cong E^{n+1} - Y$ implies* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

If we use Theorem 3.1 in place of [**ISV**, Theorem 4], we can prove stronger versions of Theorems A, B and C of [**ISV**]. The stronger versions of Theorems A and C are stated as Propositions 3.2 and 3.3 below. The main improvement is that the unnatural codimension 4 hypothesis in the converses of the original theorems is replaced by the more natural codimension 3 hypothesis.

(3.2) PROPOSITION. *Let X and Y be r-shape connected continua in S^n of fundamental dimension at most k and satisfying ILC, where*

$$n \geq \max(2k + 2 - r, k + 3, 5).$$

Then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *if and only if* $S^n - X \cong S^n - Y$.

(3.3) PROPOSITION. *Let X and Y be $(r-1)$-shape connected, pointed r-movable continua in S^n of fundamental dimension at most k and satisfying ILC, where $n \geq \max(2k+2-r, k+3, 5)$. Then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *implies* $S^n - X \cong S^n - Y$. *The converse holds if X and Y are 1-shape connected.*

(3.4) OBSERVATION. Suppose X and Y are compacta in S^n. It follows from [**No**, Proposition 6.1] that $S^n - X \cong S^n - Y$ implies $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, provided X and Y are r-shape connected and $r \geq \max(\frac{1}{2}\mathrm{Fd}(X), 1)$. This represents an improvement over Proposition 3.2 in case k is in the metastable range $k \geq \frac{2}{3}n - 1$.

PROOF OF THEOREM 3.1. Let $M_1 \supset M_2 \supset M_3 \supset \cdots$ be a sequence of PL manifold neighborhoods of X in E^n such that $M_{j+1} \subset \mathrm{Int}\, M_j$ for each j and $X = \bigcap_{j=1}^{\infty} M_j$. Let $N_j = M_j \times [-1/j, 1/j] \subset E^n \times E^1 = E^{n+1}$ and let $h\colon E^{n+1} - X \to E^{n+1} - Y$ be a homeomorphism. We may assume that h induces a homeomorphism of the quotient space E^{n+1}/X onto E^{n+1}/Y and so $N_j' = E^{n+1} - h(E^{n+1} - N_j)$ is a compact topological manifold neighborhood of Y and $Y = \bigcap_{j=1}^{\infty} N_j'$. For each j, let $p_j\colon N_j \to M_j \times \{1/j\}$ denote the natural projection and let $f_j\colon N_j \to N_j'$ be defined by $f_j = h \circ p_j$.

The proof on pp. 214 and 215 of [**ISV**] shows that, for each q, f_j induces an isomorphism

$$(f_j)_\#\colon H_q(N_j) \xrightarrow{\cong} H_q(N_j').$$

The proof on those pages also shows that $\{f_j\}_{j=1}^{\infty}$ determines a level preserving shape morphism from X to Y. By the homological version of the Whitehead Theorem for shape theory [**MS**, Theorem 13, p. 155], this shape map is a shape equivalence. □

As remarked in §2, for $n \geq 5$ there can be no knot exterior of fundamental dimension $[\frac{n}{2}]$ in S^n. Our last two theorems generalize that result; they indicate that if $S^n - X \cong S^n - Y$ and $\mathrm{Fd}(X) = [\frac{n}{2}]$, then it must also be expected that $\mathrm{Fd}(Y) = [\frac{n}{2}]$. The proof of the next theorem follows the lines of Chapman's proof of his original finite dimensional complement theorem [**Ch**$_2$, Theorem 1(b)].

(3.5) THEOREM. *Suppose X and Y are continua in E^{2k+1}, $k \neq 2$. If*

(1) $\mathrm{Fd}(X) \leq k$,
(2) $\mathrm{Fd}(Y) < k$,
(3) *X and Y satisfy the inessential loops condition, and*
(4) *X is pointed 1-movable,*

then $E^{2k+1} - X \cong E^{2k+1} - Y$ *implies* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

PROOF. By the embedding theorem of Husch and Ivanšić [**HI**$_1$, Theorem 3.2(a)], there is a copy of X in $E^{2k} \subset E^{2k+1}$. We can also embed Y up to shape in E^{2k} since $\mathrm{Fd}(Y)$ is in the trivial range relative to $2k$. Theorem C of [**ISV**] allows us to assume that X and Y themselves are subsets of E^{2k}. The proof of the theorem in [**HI**$_1$] shows that the copies of X and Y we are working with are in standard position in E^{2k}; i.e., there exist neighborhoods $M_1 \supset M_2 \supset M_3 \supset \cdots$ of X in E^{2k} such that $M_{j+1} \subset \mathrm{Int}\, M_j$ and each M_j is a compact PL manifold with a spine K_j such that $\dim K_j \leq k$ and neighborhoods $M_1' \supset M_2' \supset M_3' \supset \cdots$ of Y in E^{2k} such that $M_{j+1}' \subset \mathrm{Int}\, M_j'$ and each M_j' is a compact PL manifold with a spine K_j' such that $\dim K_j' \leq k-1$.

Let $N_j = M_j \times [-1/j, 1/j] \subset E^{2k+1}$ and let $N_j' = M_j' \times [-1/j, 1/j]$. Let $p_j\colon N_j \to M_j \times \{1/j\}$ and $p_j'\colon N_j' \to M_j' \times \{1/j\}$ be the natural projections. Choose a homeomorphism $h\colon E^{2k+1} - X \to E^{2k+1} - Y$. We may assume that h induces a homeomorphism of the quotient space E^{2k+1}/X onto E^{2k+1}/Y. Define $f_j\colon N_j \to E^{2k+1}$ and $f_j'\colon N_j' \to E^{2k+1}$ by $f_j = h \circ p_j$ and $f_j' = h^{-1} \circ p_j'$. Then $\{f_j\}_{j=1}^{\infty}$ and $\{f_j'\}_{j=1}^{\infty}$ induce shape morphisms $\mathbf{f}\colon X \to Y$ and $\mathbf{f}'\colon Y \to X$.

The proof is completed by showing that $\mathbf{f}$ and $\mathbf{f}'$ are inverse shape morphisms. We will show that $\mathbf{f}' \circ \mathbf{f}$ is equivalent to the identity and omit the other proof since it is similar. It suffices to prove the following.

(**) *If U is a neighborhood of X and i, j are integers such that $h^{-1}(N_i' - Y) \subset U$ and $f_j(N_j) \subset N_i'$, then $f_i' \circ f_j\colon N_j \to U$ is homotopic to the inclusion $N_j \hookrightarrow U$.*

Let $r_j\colon N_j \to K_j$ be the end of a strong deformation retraction. Now $p_i' \simeq \mathrm{id}_{N_i'}$ and the track of the homotopy from $p_i'(h(p_j(K_j)))$ to $h(p_j(K_j))$ is at most $(k+1)$-dimensional. Thus, by general position, we may assume that the track of the homotopy misses K_l' for some $l > i$ and hence that the homotopy takes place in $N_i' - Y$. We therefore have

$$\begin{aligned} f_i' \circ f_j &= h^{-1} \circ p_i' \circ h \circ p_j \\ &\simeq h^{-1} \circ p_i' \circ h \circ p_j \circ r_j \\ &\simeq h^{-1} \circ h \circ p_j \circ r_j \\ &= p_j \circ r_j \\ &\simeq \text{inclusion}, \end{aligned}$$

where all the homotopies take place in U. □

(3.6) REMARK. The proof of Theorem 3.5 actually shows the following: If X and Y are compacta in standard position in $E^{n-1} \subset E^n$, then $E^n - X \cong E^n - Y$ implies $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ as long as $\mathrm{Fd}(X) + \mathrm{Fd}(Y) \leq n-2$.

An obvious way to try to generalize the proof of Theorem 3.1 to compacta which are not 1-shape connected is to attempt to apply the argument to some kind of covering space. The proof of our last theorem illustrates some of the difficulties involved.

(3.7) THEOREM. *Suppose X and Y are continua in S^{2k+2}, $k \geq 2$. If*

(1) $\mathrm{Fd}(X) \leq k+1$,
(2) $\mathrm{Fd}(Y) < k+1$,

(3) *X and Y satisfy the inessential loops condition, and*
(4) pro-$\pi_1(X) \cong \mathbf{Z}$,
then $S^{2k+2} - X \cong S^{2k+2} - Y$ *implies* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

PROOF. Since X and Y have homeomorphic complements and both have codimension three, it is not hard to see that pro-$\pi_1(Y) \cong$ pro-$\pi_1(X)$. By [**HI**$_3$, Theorem 2], there exists a continuum Y' such that $\mathrm{Sh}(Y) = \mathrm{Sh}(Y')$ and

$$Y' = \varprojlim\{K_1 \xleftarrow{\phi_1} K_2 \xleftarrow{\phi_2} K_3 \leftarrow \cdots\}$$

where each K_j is a compact polyhedron of dimension $\leq k$ and ϕ_j induces an isomorphism $\pi_1(K_{j+1}) \to \pi_1(K_j)$. Since $\pi_1(K_1) \cong \mathbf{Z}$, it is easy to construct a map $f\colon K_1 \to S^1$ which induces an isomorphism on π_1. The map $f\colon K_1 \to S^1$ determines a map $\hat{f}\colon Y' \to S^1$ which induces an isomorphism on pro-π_1.

Let $S^1 \subset S^{2k+1}$ be the standard embedding and let Σ be a locally flat, unknotted PL $(2k-1)$-sphere which is a spine of $S^{2k+1} - S^1$. We have $\dim Y' \leq k$, so the map $\hat{f}\colon Y' \to S^1 \subset S^{2k+1} - \Sigma$ can be approximated by a 1-LCC embedding $e\colon Y' \to S^{2k+1} - \Sigma$. Make the approximation close enough so that $e(Y') \hookrightarrow S^{2k+1} - \Sigma$ induces an isomorphism on pro-π_1. By [**Ve**$_1$], $S^{2k+2} - Y \cong S^{2k+2} - e(Y')$, so we may assume for the rest of the proof that Y is 1-LCC embedded in $S^{2k+1} - \Sigma \subset S^{2k+2}$ and that $Y \hookrightarrow S^{2k+1} - \Sigma$ induces a pro-π_1 isomorphism.

By [**HI**$_3$, Proposition 1], there is a sequence $M_1 \supset M_2 \supset M_3 \supset \cdots$ of compact PL manifold neighborhoods of Y in S^{2k+1} such that $M_{j+1} \subset \mathrm{Int}\, M_j$ and $M_{j+1} \hookrightarrow M_j$ induces a π_1 isomorphism for each j. Since S^{2k+1} is collared in S^{2k+2}, we may assume that $S^{2k+1} \times [-1,1] \subset S^{2k+2}$ and identify S^{2k+1} with $S^{2k+1} \times \{0\}$. Define $N_j = M_j \times [-1/j, 1/j] \subset S^{2k+2}$. We may also identify S^{2k+2} with the suspension of S^{2k+1}. Let Σ' denote the suspension of Σ in S^{2k+2} and make sure that $N_1 \cap \Sigma' = \varnothing$.

Let $h\colon S^{2k+2} - Y \to S^{2k+2} - X$ be a homeomorphism. Consider the sequence

$$\begin{aligned}\pi_l(S^{2k+2} - h(\Sigma')) &\xleftarrow{\alpha} \pi_l(S^{2k+2} - (h(\Sigma') \cup X)) \\ &\xleftarrow{h_*} \pi_l(S^{2k+2} - (\Sigma' \cup Y)) \xrightarrow{\beta} \pi_l(S^{2k+2} - \Sigma')\end{aligned}$$

in which α and β are inclusion induced and h_* is the isomorphism induced by $h|S^{2k+2} - (\Sigma' - Y)$. Since $\mathrm{Fd}(X) \leq k+1$, α is an isomorphism for $l \leq k-1$ and an epimorphism for $l = k$. Similarly, the fact that $\mathrm{Fd}(Y) \leq k$ implies that β is an isomorphism for $l \leq k$ and an epimorphism for $l = k+1$. Thus $\pi_1(S^{2k+2} - h(\Sigma')) \cong \mathbf{Z}$ and $\pi_l(S^{2k+2} - h(\Sigma')) = 0$ for $2 \leq l \leq k$. Hence Levine's Theorem [**Le**] implies that $h(\Sigma')$ is unknotted.

Define $N_j' = S^{2k+2} - h(S^{2k+2} - N_j)$. As in the proof of Theorem 3.1, there are naturally defined maps $f_j\colon N_j \to N_j'$ which, in turn, determine a shape map from Y to X. The proof on pp. 214 and 215 of [**ISV**] shows that this map is a shape equivalence provided each f_j is a homotopy equivalence. By a theorem of Whitehead [**Wh**, Theorem 3], it is enough to show that each f_j induces an isomorphism on π_1 and an isomorphism on the homology of the universal covers.

Since X and Y have codimension 3, it is easy to see that f_j induces an isomorphism on π_1. Let $p\colon U \to S^{2k+2} - \Sigma'$ and $q\colon V \to S^{2k+2} - h(\Sigma')$ denote the universal covers. We have set things up so that $N_j \hookrightarrow S^{2k+2} - \Sigma'$ and $N_j' \hookrightarrow S^{2k+2} - h(\Sigma')$

are π_1-equivalences. Thus $p^{-1}(N_j)$ and $q^{-1}(N_j')$ are the universal covers of N_j and N_j' respectively and f_j is covered by a map $\tilde{f}_j\colon p^{-1}(N_j) \to q^{-1}(N_j')$. Notice that both the universal covers U and V are contractible and so is $p^{-1}(S^{2k+1} - \Sigma)$. Also, the homeomorphism $h|S^{2k+2} - (\Sigma' \cup Y)\colon S^{2k+2} - (\Sigma' \cup Y) \to S^{2k+2} - (h(\Sigma') \cup X)$ lifts to a homeomorphism $\tilde{h}\colon U - p^{-1}(Y) \to V - q^{-1}(X)$. Thus the argument on p. 215 of [**ISV**] applies to show that $\tilde{f}_j$ is a homology equivalence. $\square$

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