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A SPECIAL CASE OF POSITIVITY

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ABSTRACT. In this note we prove a special case of positivity of Serre's Conjecture on intersection multiplicity of modules [**S**]. The conjecture can be stated as follows.

Let R be a regular local ring and let M and N be two finitely generated modules over R such that $l(M \otimes N) < \infty$. Then $\chi(M,N) = \sum_{i=0}^{\dim R}(-1)^i l(\mathrm{Tor}_i^R(M,N)) \geq 0$, the sign of inequality holds if and only if $\dim M + \dim N = \dim R$.

Serre proved the conjecture in the equicharacteristic and in the unramified case. Recently P. Roberts [**R**] and H. Gillet and C. Soulé [**H-G**] proved independently the vanishing part, i.e. $\chi(M,N) = 0$ when $\dim M + \dim N < \dim R$ in more generality. The positivity part, i.e. $\chi(M,N) > 0$ when $\dim M + \dim N = \dim R$ is still very much an open question.

We write $R = V[[x_1,\ldots,x_n]]/(f)$, V a complete discrete valuation ring, p a generator of the maximal ideal of V, $p \in m^2$ where m is the maximal ideal of R and $f \in m - m^2$. We divide the whole problem into three parts:

1. $pM = 0$, $pN = 0$. This case was proved by Malliavin-Brameret [**M**].
2. p is a non-zero-divisor on M and p is nilpotent on N.
3. p is a non-zero-divisor on both M and N.

The theorem which we are going to prove is the following

THEOREM. *Let R be a regular local ring. Let M and N be two finitely generated modules over R such that*

(i) *M is Cohen-Macaulay.*

(ii) *$l(M \otimes N) < \infty$ and $\dim M + \dim N = \dim R$.*

(iii) *$p^t N = 0$ for some integer t and p is a non-zero-divisor on M.*

Then $\chi(M,N) > 0$.

The above theorem was already proved by the author in the case when $\dim R \leq 5$ in [**D2**]. The vanishing theorem of Roberts (Gillet and Soulé) and the techniques developed by the author in [**D1**] now make it possible to prove the above version.

PROOF OF THE THEOREM. We divide the proof into two steps.

Step 1. Let R be a Gorenstein local ring of characteristic $p > 0$. Let M be a module of finite projective dimension and let N be any other module over R such that $l(M \otimes N) < \infty$ and $\dim M + \dim N \leq \dim R$.

Let $f: R \to R$ be the Frobenius map, i.e. $f(x) = x^p$. We denote by f_R^n the bialgebra R, having the structure of an R-algebra from the left by f^n and from

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the right by the identity map, i.e. if $\alpha \in R$, $x \in f_R^n$, $\alpha x = \alpha^{p^n} x$, and $x\alpha = x\alpha$. We assume $K = R/m$, where m is the maximal ideal of R, is perfect. (This assumption is not at all restrictive with respect to generalized type of intersection multiplicity conjectures.) We denote by $F^n(M)$ the object $M \otimes f_R^n$. We define $\chi_\infty(M,N) = \lim_{n\to\infty} \chi(F^n(M),N)/p^{n\cdot \operatorname{codim} M}$. The following properties of χ_∞ were proved in [**D1**].

1. If $\dim M + \dim N < \dim R$, then $\chi_\infty(M,N) = 0$ (Corollary 1, p. 437).
2. If M is Cohen-Macaulay, then

$$\chi_\infty(M,N) = \lim_{n\to\infty} l(F^n(M)\otimes N)/p^{n\cdot \operatorname{codim} M}$$

and this is a positive integer if R is a complete intersection.

3. $\chi_\infty(M,N) = \chi(M,N)$ if the vanishing conjecture holds for every pair (M,T) with $l(M\otimes T) < \infty$ and $\dim M + \dim T < \dim R$ (this assertion follows easily from Proposition 1.2 of [**D1**]).

Step 2. Under the hypothesis in our theorem, since χ is additive, we can assume $pN = 0$. Since p is a non-zero-divisor on both R and M and $pN = 0$, we have

(i) $\chi^R(M,N) = \chi^{R/pR}(M/pM,N)$.

(ii) $\text{P.d.}_{R/pR}\ M/pM < \infty$ and $\chi^{R/pR}(M/pM,T) = 0$, where

$$\dim M/pM + \dim T < \dim R/pR$$

(since this implies $\dim M + \dim T < \dim R$, and $\chi(M,T) = 0$ [**G-S**, **R**]).

(iii) M/pM is Cohen-Macaulay over R/pR with Ch. $R/pR = p > 0$.

We denote R/pR by $\overline{R}$. We have by (ii) and (3) of Step 1

$$\chi_\infty^{\overline{R}}(\overline{M},N) = \chi^{\overline{R}}(\overline{M},N) = \chi^R(M,N).$$

Moreover by (iii) and (2) of Step 1

$$\chi_\infty^{\overline{R}}(\overline{M},N) = \lim_{n\to\infty} l(F^n(\overline{M})\otimes N)/p^{n\cdot \operatorname{codim} \overline{M}}.$$

Hence $\chi^R(M,N) > 0$.

REMARK. Unfortunately, χ_∞ fails to behave like a "multiplicity function" over R for pairs (M,N) with P.d. $M < \infty$, $l(M\otimes N) < \infty$, $\dim M + \dim N = \dim R$ when M is *not* Cohen-Macaulay. This was pointed out in [**D-H-M**]. The counterexample discussed there gives rise to a module M with P.d. $M < \infty$, $\dim M = 1$, depth $M = 0$ and another module N such that $\chi_\infty(M,N)$ is negative.

REFERENCES

[**D1**] S. P. Dutta, *Frobenius and multiplicity*, J. Algebra **85** (1983).

[**D2**] ——, *Generalized intersection multiplicities of modules*, Trans. Amer. Math. Soc. **276** (1983).

[**D-H-M**] S. P. Dutta, M. Hochster, and J. E. McLaughlin, *Modules of finite projective dimension with negative intersection multiplicities*, Invent. Math. **79** (1985), 253–291.

[**G-S**] H. Gillet and C. Soulé, *K theorie et nullite des multiplicites d'intersection*, C. R. Acad. Sci. Paris Ser. I **300** (1985), 71–74.

[**H**] M. Hochster, *Conference on commutative algebra*, Lecture Notes in Math., Vol. 311, Springer-Verlag, Berlin and New York, 1973, pp. 120–152.

[M] M. P. Malliavin-Brameret, *Une remarque sur les anneaux locaux reguliers*, Sém. Dubreil-Pisot Algèbre et Théorie des Nombres, 24 année, 1970/71, No. 13.
[R] P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bull. Amer. Math. Soc. **13** (1985).
[S] J.-P. Serre, *Algèbre locale multiplicités*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin, Heidelberg, and New York, 1957/58.

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