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POLARITY IN A COMPLETELY DISTRIBUTIVE COMPLETE LATTICE

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ABSTRACT. We introduce p-bases in completely distributive complete polarity lattices and give a procedure for generating these lattices by p-bases.

1. Introduction. As in [**1**], the object of this paper is to lay a lattice-theoretic foundation of treating local properties of the generalized algebra of topology i.e., the algebra of topology [**3**] with polarity instead of Boolean complement. $p^{\sim}$ in Definition 3.2 is introduced to overcome the difficulty in the unified treatment of Boolean complement and non-Boolean polarity and Proposition 3.2 can be regarded as a generalization of Boolean property.

We first examine polarity by means of quasi-atoms introduced in [**1**]. And then we introduce p-bases in completely distributive complete polarity lattices and give a procedure for the generation of these lattices by p-bases.

2. Preliminaries. Let $(L, \leq)$ be a completely distributive complete lattice. We shall use a, b, c to denote its elements and A, B, C its subsets.

DEFINITION 2.1. $a \prec b$ iff for every A with $\bigvee A = b$ there exists $c \in A$ such that $a \leq c$.

DEFINITION 2.2 [**1**]. $p \in \Delta(L) \subseteq L$ where $\Delta(L) = \{\bigwedge_{a \prec b} b | a \neq 0, \bigvee_{a \not\leq c} C < 1\}$ is called a quasi-atom.

We shall use p, q, r to denote elements of $\Delta(L)$ and P, Q, R its subsets.

DEFINITION 2.3 [**1**]. (1) $Q \subseteq \Delta(L)$ is called a base of $(L, \leq)$ iff $a = \bigvee\{p | p \in Q, p \prec a\}$ for every $a \in L$.

(2) $p \in \Delta(L)$ has the $\wedge$-property iff for every a, $b \in L$, that $p \prec a$ and $p \prec b$ implies $p \prec a \wedge b$.

(3) A base Q of which every element is $\vee$-irreducible and has the $\wedge$-property is called a s-base.

(4) $(L, \leq)$ is standard iff it has a s-base.

(5) A l-set of a base Q is a subset R of Q satisfying (1) $p \in R$ implies that there exists q such that $p \prec q \in R$ and (2) $p \prec q \in R$ implies $p \in R$ for every $p \in Q$.

(6) For $a \in L$, $R_a = \{p | p \prec a, p \in Q\}$.

The set of all l-sets of a base Q will be denoted by $L(Q)$. Evidently $R_a \in L(Q)$ for every $a \in L$. Let $\Phi\colon Q \to L(Q)$ be defined as $\Phi(p) = R_p$. We shall use $I(L)$ to denote the set of all completely $\vee$-irreducible elements of L and $A(L)$ the set of all its atoms.

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3. Polarity.

DEFINITION 3.1 [2]. A polarity $'$ in a lattice is involutary antiautomorphism.

Throughout the remaining part of this paper $(L, \leq, ')$ will denote a completely distributive complete polarity lattice (in short p-lattice). By a Boolean p-lattice we mean a p-lattice with Boolean complement as polarity.

DEFINITION 3.2. $p^{\sim} = \bigwedge_{q \not\prec p'} q$ for $p, q \in \Delta(L)$.

LEMMA 3.1. $q \prec p'$ implies $p \prec q'$ for every $p, q \in \Delta(L)$.

PROOF. By [1, Proposition 2.1] $p = \bigwedge_{p \prec a} a$, hence $p' = \bigvee_{p \prec a} a'$. Since $q \prec p'$ implies that there exists a with $p \prec a$ such that $q \leq a'$, it follows that $p \prec q'$ by [1, Lemma 2.1(5)]. Q.E.D.

LEMMA 3.2. $p^{\sim} \in \Delta(L)$.

PROOF. Since $p \neq 0$, there exists $r \in \Delta(L)$ such that $r \prec p'$ by [1, Lemma 3.2(2)] noting that $\Delta(L)$ itself is a base. It follows that $p^{\sim} \leq r$, which implies $\bigvee_{p^{\sim} \not\leq a} a \leq \bigvee_{r \not\leq b} b < 1$. By Lemma 3.1 $p^{\sim\prime} = \bigvee_{p \not\prec q'} q'$, so $p \not\prec p^{\sim\prime}$, hence $p^{\sim} \neq 0$. Since $\bigwedge_{p^{\sim} \prec a} a \leq \bigwedge_{q \not\prec p'} \bigwedge_{q \prec b} b = \bigwedge_{q \not\prec p'} q$ by [1, Proposition 2.1], $p^{\sim} = \bigwedge_{p^{\sim} \prec a} a \in \Delta(L)$. Q.E.D.

PROPOSITION 3.1. *$p^{\sim} \not\prec p'$ and for every $q \in \Delta(L)$, $q \not\prec p'$ implies $p^{\sim} \leq q$.*

PROOF. In the proof of Lemma 3.2 we have already shown that $p \not\prec p^{\sim\prime}$ and so $p^{\sim} \not\prec p'$ by Lemma 3.1. The remaining part is trivial. Q.E.D.

PROPOSITION 3.2. *$p \not\leq a'$ iff $p^{\sim} \prec a$ for every $p \in \Delta(L)$, $a \in L$.*

PROOF. $p \not\leq a'$ iff $a \not\leq p'$, which holds iff there exists $q \in \Delta(L)$ such that $q \prec a$ and $q \not\prec p'$ by [1, Lemma 3.2(2)], which holds iff $p^{\sim} \prec a$ by Proposition 3.1 and [1, Lemma 2.1(2), Lemma 3.2(1)]. Q.E.D.

PROPOSITION 3.3. (1) *$p^{\sim\sim} = p$.* (2) *$p \prec q$ implies $q^{\sim} \prec p^{\sim}$.*

PROOF. (1) Since $p^{\sim} \not\prec p'$, $p \not\prec p^{\sim\prime}$, which implies $p^{\sim\sim} \leq p$. Futhermore $p^{\sim\sim} \not\prec p^{\sim\prime}$, so $p^{\sim} \not\prec p^{\sim\sim\prime}$, hence $p \leq p^{\sim\sim}$.

(2) Suppose $p \prec q$, then $p^{\sim} \not\leq q'$, hence $q \not\leq p^{\sim\prime}$ and so $q^{\sim} \prec p^{\sim}$. Q.E.D.

From above we see that $p^{\sim} = \min\{q | q \not\prec p', q \in \Delta(L)\}$ and $\sim: \Delta(L) \to \Delta(L)$ is a mapping satisfying Propositions 3.2 and 3.3. If $(L, \leq, ')$ is a Boolean p-lattice, then $\Delta(L) = A(L)$, the binary relation $\prec$ is the same as $\leq$ and $p^{\sim} = p$. We shall need the following lemmas.

LEMMA 3.3. *$p \in \Delta(L)$ is $\vee$-irreducible iff $p^{\sim}$ has the $\wedge$-property.*

PROOF. For $p \leq a \vee b$ iff $p^{\sim} \not\prec a' \wedge b'$. Q.E.D.

LEMMA 3.4. *If $^{\sim}: Q \to Q$ where $Q \subseteq \Delta(L)$ satisfies Proposition 3.2 and Proposition 3.3(1), then $p^{\sim} = \min\{q | q \not\prec p', q \in \Delta(L)\}$.*

PROOF. Since $p \leq p$, it follows that $p^{\sim} \not\prec p'$. On the other hand if $q \in \Delta(L)$ and $q \not\prec p'$, then $p \not\prec q'$, hence $p^{\sim} \leq q$. Q.E.D.

4. p-bases.

DEFINITION 4.1. A p-base of $(L, \leq, ')$ is a base Q which is closed under $^{\sim}$.

PROPOSITION 4.1. *Any p-lattice has a p-base.*

PROOF. By [1, Proposition 2.2] $\Delta(L)$ itself is a base and p-base. Q.E.D.

PROPOSITION 4.2. (1) *Any standard p-lattice has a p-s-base.*

(2) *Any p-lattice isomorphic to a complete ring of sets has a p-base $I(L)$.*

(3) *Any Boolean p-lattice has a p-base $A(L)$.*

PROOF. (1) $Q = \{p | p \in \Delta(L),\ p$ is $\vee$-irreducible and p has the $\wedge$-property$\}$ is evidently a s-base and by Lemma 3.3 and Proposition 3.3(1) a p-base.

(2) By Proposition 3.3 (2) $p \prec p$ implies $p^{\sim} \prec p^{\sim}$ and so $p^{\sim} \in I(L)$ by [**1**, Lemma 2.1(4)].

(3) For $p^{\sim} = p \in A(L)$. Q.E.D.

For R_p, $L(P)$ and Φ in the following propositions see §2 of this paper.

PROPOSITION 4.3. *If Q is a p-base, then*

(1) $\prec_Q \circ \prec_Q = \prec_Q$ *where $\prec_Q$ is the restriction of $\prec$ on Q.*

(2) *For every $p,\ q,\ r,\ s \in Q$, $q \prec r \in \bigcap_{p \prec s} R_s$ implies $q \prec p$.*

(3) *For every $p \in Q$ there exists $q,\ r \in Q$ such that $q \prec p \prec r$.*

(4) *For every $p,\ q \in Q$, $R_p = R_q$ implies $p = q$.*

(5) *For every $p,\ q,\ r \in Q$, that $R_p \subseteq R_q$ and $q \prec r$ implies $p \prec r$.*

(6) *For every $p,\ q \in Q$, $p \prec q$ implies $q^{\sim} \prec p^{\sim}$.*

(7) *For every p, $p^{\sim\sim} = p$.*

If Q is a p-s-base, then

(8) *For every $p,\ q,\ r \in Q$, $p \in R_q \cap R_r$ implies that there exists $s \in Q$ such that $p \prec s$ and $s \in R_q \cap R_r$.*

(9) *R_p is $\vee$-irreducible in $L(Q)$ ordered by inclusion.*

PROOF. It follows from [**1**, Proposition 3.3] and Proposition 3.3. Q.E.D.

Let $P_{i,j,\dots}$ be any nonvoid set on which there are defined a binary relation $\prec$ and a mapping $\sim$ satisfying the conditions (i), $(j), \dots$ in Proposition 4.3.

PROPOSITION 4.4. (1) *$L(P_{1,2,6,7})$ ordered by inclusion is a p-lattice.*

(2) *$\Phi(P_{1,2,\dots,7})$ is a p-base of $L(P_{1,2,\dots,7})$ and Φ is an isomorphism from $P_{1,2,\dots,7}$ onto $\Phi(P_{1,2,\dots,7})$.*

(3) *$\Phi(P_{1,2,\dots,9})$ is a p-s-base.*

PROOF. (1) By [**1**, Proposition 4.1(1)] $L(P_{1,2,6,7})$ is a completely distributive complete lattice. Define $'$ as follows: $R' = \{p |$ there exists q such that $p \prec q$ and $q^{\sim} \notin R\}$ for $R \in L(P_{1,2,6,7})$ and $p,\ q \in P_{1,2,6,7}$. That $R \leq S$ implies $S' \leq R'$ is trivial. Since $p \in R$ iff there exists q such that $p \prec q$ and $R_q \subseteq R$, and since $R_q \subseteq R$ iff for every s, $q^{\sim} \prec s$ implies $s^{\sim} \in R$, which is equivalent to $q^{\sim} \notin R'$, it follows that $p \in R$ is equivalent to $p \in R''$, hence $R'' = R$. Thus $'$ is a polarity.

(2) By [**1**, Proposition 4.1(1)-(4)] $\Phi(P_{1,2,\dots,7})$ is a base, Φ is a bijection and both Φ and Φ^{-1} are $\prec$-preserving. Define $R_p^{\sim} = R_{p^{\sim}}$. Then $R_p \prec S'$ iff $p \in S'$, which holds iff there exists q such that $p \prec q$ and $q^{\sim} \notin S$, which holds iff there exists $q^{\sim}$ such that $q^{\sim} \prec p^{\sim}$ and $q^{\sim} \notin S$, which is equivalent to $R_p^{\sim} = R_{p^{\sim}} \nsubseteq S$. Evidently $R_p^{\sim\sim} = R_p$. By Lemma 3.4 $R_p^{\sim} = \min\{R | R \not\prec R_p',\ R \in \Delta(L(P_{1,2,\dots,7}))\}$ and so $\Phi(P_{1,2,\dots,7})$ is a p-base. Since $\Phi(p^{\sim}) = R_{p^{\sim}} = R_p^{\sim} = \Phi(p)^{\sim}$ Φ is an isomorphism.

(3) By [**1**, Proposition 4.1(5),(6)] and (2) $\Phi(P_{1,2,\dots,9})$ is a p-s-base. Q.E.D.

The above proposition can be used to construct new p-lattices from old ones.

REFERENCES

1. Z. Deng, *Quasi-atoms and complete distributivity*, Proc. Amer. Math. Soc. **103** (1988), 365–370
2. M. Kamara, *Zur konstraktion Vollstandiger Polaritätsverbande*, J. Reine Angew. Math. **299/300** (1978), 280–289.
3. J. C. C. McKinsey and A. Tarski, *The algebra of topology*, Ann. of Math. (2) **45**(1944), 141–191.

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