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MULTIPLE NONTRIVIAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We give a condition for a semilinear elliptic equation to have two nontrivial solutions. Our condition does not demand any differentiability of the nonlinear term.

1. Introduction. Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $g\colon R \to R$ be a continuous mapping such that $g(0) = 0$. We study the boundary value problem of the form

$$\Delta u + g(u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{1}$$

Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ denote the eigenvalues of the selfadjoint realization in $L^2(\Omega)$ of $-\Delta$ with the boundary condition. In [**2**], Ambrosetti and Mancini proved that if $g \in C^2$, $sg''(s) > 0$ for all $s \neq 0$ and

$$\lambda_{k-1} < g'(0) < \lambda_k < g(\pm\infty) = \lim_{s\to\pm\infty} g'(s) < \lambda_{k+1}, \quad \text{for some } k \geq 1, \tag{2}$$

then the problem (1) has exactly two nontrivial solutions. Recently, Ahmad [**1**] proved that if $g \in C^1$ and

$$0 < g'(0) < \lambda_1 < \lim_{|t|\to\infty} g(t)/t < \lambda_2, \tag{3}$$

then the problem (1) has at least two nontrivial solutions.

In the present paper, we consider the case

$$\lambda_{k-1} < b_* \leq b^* < \lambda_k < a_* \leq a^* < \lambda_{k+1}, \quad \text{for some } k \geq 1 \tag{4}$$

where $a^* = \sup_{t\neq 0} g(t)/t$, $a_* = \liminf_{|t|\to\infty} g(t)/t$, $b^* = \limsup_{|t|\to 0} g(t)/t$, and $b_* = \inf_{t\in R} g(t)/t$. In assumption (4), we have implicitly supposed λ_k is single. Our method is similar to that employed in [**5**], and does not demand that g is differentiable or $\lim_{t\to\pm\infty} g(t)/t$ exists.

THEOREM. *If* (4) *is satisfied, then the problem* (1.1) *has at least two nontrivial solutions.*

REMARK. Our result is a partial extension of Theorem 1.2 of [**2**] and also Theorem 1 of [**1**]. In fact, it is easy to see that (4) holds if $sg''(s) > 0$ for all $s \neq 0$ and g satisfies (2). It is also obvious that if (3) holds and $0 < g(t)/t < \lambda_2$ for $t \neq 0$, then (4) is satisfied with $k = 1$ and $\lambda_0 = 0$. Our argument can be applied to a more general situation (e.g., $-\Delta$ can be replaced by a more general elliptic operator).

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2. Proof of Theorem. In the following, we write L^2, H_0^1 and H^{-1} instead of $L^2(\Omega)$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, respectively. We denote by $\|\cdot\|$, $|\cdot|$ the norms of H_0^1 and L^2, respectively, the pairing between H_0^1 and H^{-1} is denoted by $\langle\cdot,\cdot\rangle$. Let H_1, H_2 and H_3 be the subspaces of L^2 spanned by the eigenspaces corresponding to the eigenvalues $\{\lambda_{k+1}, \lambda_{k+2}, \dots\}$, $\{\lambda_k\}$, and $\{\lambda_1, \lambda_2, \dots, \lambda_{k-1}\}$, respectively. Then H_1, H_2 and H_3 are orthogonal in L^2. Let ϕ be a normalized eigenfunction corresponding to λ_k. Then $\phi \in L^\infty(\Omega)$ and $H_2 = \{k\phi\colon k \in R\}$. We denote by P_1, P_2 and P_3 the projections from L^2 onto H_1, H_2 and H_3, respectively. Suppose that g satisfies the condition (4). Then there exist positive constants α, β, ρ, and δ such that

$$\alpha > \lambda_k, \qquad g(t)/t \geq \alpha \quad \text{for all } t \text{ with } |t| \geq \delta, \tag{5}$$

and

$$\beta < \lambda_k, \qquad g(t)/t \leq \beta \quad \text{for all } t \text{ with } 0 < |t| \leq \rho. \tag{6}$$

Let $L = -\Delta$. For each $u, v \in H_0^1$, we set

$$\langle Tu, v\rangle = \langle L(u - 2(P_2+P_3)u) - g(u - 2(P_2+P_3)u), v\rangle. \tag{7}$$

Then we can see that $u - 2(P_2+P_3)u$ is a solution of (1) if and only if $Tu = 0$. So we will show the existence of $u \in H_0^1$ satisfying $Tu = 0$ by making use of an existence result for pseudo-monotone mappings. Let K be a closed convex subset of a reflexive Banach space E. We denote by ∂K and $\operatorname{int} K$ the sets of boundary points and interior points of K, respectively. Let T be a mapping from K into the dual space E' of E. Then T is said to be pseudo-monotone if T satisfies the following condition:

(*) If $\{u_n\} \subset K$ is a sequence such that u_n converges weakly to u and $\limsup\langle Tu_n, u_n - u\rangle \leq 0$, then $(Tu, u - z) \leq \liminf(Tu_n, u_n - z)$ for each $z \in K$.

The following result is crucial for our argument.

THEOREM A. *Let K be a closed convex subset of E with nonempty interior, and $T\colon K \to E'$ be a pseudo-monotone mapping such that*

(P) *for each $z \in \partial K$, there exists $x \in \operatorname{int} K$ such that*

$$\langle Tz, z - x\rangle \geq 0. \tag{8}$$

Then there exists $x_0 \in K$ such that $Tx_0 = 0$.

Theorem A is a simple version of Theorem 7.8 of Browder [**3**] (see also Theorem 0 of [**5**]). It can be proved by the same argument as in the proof of Theorem 1 of [**4**], so we omit the proof.

To apply Theorem A, we need the following three lemmas.

LEMMA 1 (CF. [**5**]). *The mapping $T\colon H_0^1 \to H^{-1}$ is pseudo-monotone.*

PROOF. Let $\{u_n\} \subset H_0^1$ be a sequence such that u_n converges to u weakly in H_0^1 and

$$\begin{aligned} &\limsup\langle Tu_n, u_n - u\rangle \\ &\quad = \limsup\langle L(u_n - 2(P_2+P_3)u_n) - g(u_n - 2(P_2+P_3)u_n), u_n - u\rangle \\ &\quad \leq 0. \end{aligned} \tag{9}$$

Since H_0^1 is compactly embedded in L^2, we have that u_n converges to u strongly in L^2. We also have that $(P_2+P_3)u_n$ converges to $(P_2+P_3)u$ strongly in L^2. Then

$$\lim\langle 2L(P_2+P_3)u_n + g(u_n - 2(P_2+P_3)u_n), u_n - u\rangle = 0,$$

and therefore the inequality (9) implies that $\limsup\langle Lu_n, u_n - u\rangle \le 0$. Then it follows that Lu_n converges to Lu weakly in H^{-1} and that $\lim\langle Lu_n, u_n\rangle = \langle Lu, u\rangle$. Thus we find that Tu_n converges to Tu weakly in H^{-1} and $\lim\langle Tu_n, u_n\rangle = \langle Tu, u\rangle$. Then we obtain that

$$\langle Tu, u - z\rangle = \lim\langle Tu_n, u_n - z\rangle \quad \text{for each } z \in H_0^1,$$

and this completes the proof.

LEMMA 2. *There exists $s > 0$ such that*

$$\langle Tu, u - 2P_2u\rangle \ge 0 \quad \textit{for all } u \in H_0^1 \textit{ with } \|P_2u\| \le s.$$

PROOF. We first choose a positive number c so small that

$$\min\{\tfrac{1}{2}(\lambda_{k+1} - a^*), (b_* - \lambda_{k-1})\}(\rho - c)^2/4 - (a^* - \lambda_k)c^2 + \tfrac{1}{2}(\lambda_{k+1} - a^*)d^2 - 2c(a^* - \beta)d > 0$$

for all $d \in R$. Since $\phi \in L^\infty$, we can choose $s > 0$ such that $\sup_{x\in\Omega}|z(x)| \le c$ for all $z \in H_2$ with $\|z\| \le s$. Let $u \in H_1^0$ with $\|P_2u\| \le s$. We set, for simplicity, $v = P_1u$, $w = P_2uz = P_3u$, and $\tilde{u} = v - w - z$. We also set $A = \{x \in \Omega\colon |\tilde{u}| > \rho\}$ and $B = \{x \in \Omega\colon |\tilde{u}| \le \rho\}$. From the definition of T, we have

$$\begin{aligned} \langle Tu, u - 2P_2u\rangle &= \langle L(v - w - z) - g(v - w - z), v - w + z\rangle \\ &\ge \lambda_{k+1}|v|^2 + \lambda_k|w|^2 - \lambda_{k-1}|z|^2 - \int_\Omega g(\tilde{u})(v - w + z)\,dx. \end{aligned} \tag{10}$$

Let $x \in A$. Then since $|w(x)| \le c$, we have that $\max\{|v(x)|, |z(x)|\} > (\rho - c)/2$. If $|z(x)| \ge |(v - w)(x)|$, then noting that $\tilde{u}(x)(v - w + z)(x) \le 0$ we find from (4) that

$$-g(\tilde{u}(x))(v - w + z)(x) \ge b_*(|z(x)|^2 - |(v - w)(x)|^2).$$

If $|z(x)| < |(v - w)(x)|$, then we have by (4) that

$$-g(\tilde{u}(x))(v - w + z)(x) \ge a^*(|z(x)|^2 - |(v - w)(x)|^2).$$

Then from the inequalities above, we find that

$$\begin{aligned} &\lambda_{k+1}|v(x)|^2 + \lambda_k|w(x)|^2 - \lambda_{k-1}|z(x)|^2 - g(\tilde{u}(x))(v - w + z)(x) \\ &\quad\ge \lambda_{k+1}|v(x)|^2 + \lambda_k|w(x)|^2 - \lambda_{k-1}|z(x)|^2 + b_*|z(x)|^2 - a^*|(v - w)(x)| \\ &\quad\ge (\lambda_{k+1} - a^*)|v(x)|^2 + (\lambda_k - a^*)|w(x)|^2 + (b_* - \lambda_{k-1})|z(x)|^2 \\ &\qquad + 2(a^* - \beta)w(x)v(x) + 2\beta w(x)v(x) \\ &\quad\ge \min\{\tfrac{1}{2}(\lambda_{k+1} - a^*), (b_* - \lambda_{k-1})\}(\rho - c)^2/4 - (a^* - \lambda_k)c^2 \\ &\qquad + (\tfrac{1}{2}(\lambda_{k+1} - a^*)|v(x)|^2 - 2c(a^* - \beta)|v(x)|) + 2\beta w(x)v(x) \\ &\quad\ge 2\beta w(x)v(x). \end{aligned} \tag{11}$$

Let $x \in B$. Then we have from (4) and (6) that

$$-g(\tilde{u}(x))(v - w + z)(x) \ge b_*|z(x)|^2 - \beta|(v - w)(x)|^2. \tag{12}$$

Then we find that

$$
\begin{aligned}
&\lambda_{k+1}|v(x)|^2 + \lambda_k|w(x)|^2 - \lambda_{k-1}|z(x)|^2 - g(\tilde{u}(x))(v - w + z)(x) \\
&\quad \geq (\lambda_{k+1} - \beta)|v(x)|^2 + (\lambda_k - \beta)|w(x)|^2 \\
&\qquad + (b_* - \lambda_{k-1})|z(x)|^2 + 2\beta w(x)v(x) \\
&\quad \geq 2\beta w(x)v(x).
\end{aligned} \tag{13}
$$

Then combining (10) with (11) and (13), we obtain that

$$\langle Tu, u\rangle \geq 2\beta\left(\int_A w(x)v(x)\,dx + \int_B w(x)v(x)\,dx\right) = 2\beta\langle w, v\rangle = 0,$$

and this completes the proof.

LEMMA 3. *There exists $r > 0$ such that $r > 2s$ and $\langle Tu, u\rangle \geq 0$ for all $u \in H_0^1$ with $\|u\| \geq r$.*

PROOF. Let λ be a positive number such that $a^* < \lambda < \lambda_{k+1}$. Then we have that $M = \min_{t\in R}\{(\lambda - b_*)t^2 - (\alpha - b_*)(t + \delta)^2\} > -\infty$. Also we have that there exists $\omega > 0$ such that $\|v\|^2 - \lambda|v|^2 \geq \omega\|v\|^2$ for all $v \in H_1$. Let $u \in H_0^1$. Let v, w, z and $\tilde{u}$ be as in Lemma 2. We put $A = \{x \in \Omega\colon |\tilde{u}(x)| \geq \delta\}$, $B = \{x \in \Omega | \tilde{u}(x)| < \delta$, $|v(x)| < |(w + z)(x)|\}$ and $C = \{x \in \Omega\colon |\tilde{u}(x)| < \delta,\ |v(x)| \geq |(w + z)(x)|\}$. Then we have

$$\langle Tu, u\rangle \geq \|v\|^2 - \lambda_k|w|^2 - \lambda_{k-1}|z|^2 - \int_\Omega g(\tilde{u})u\,dx. \tag{14}$$

From the conditions (4) and (5) we find that for each $x \in A$,

$$-g(\tilde{u}(x))u(x) \geq \alpha|(w + z)(x)|^2 - a^*|v(x)|^2.$$

Also we have by (4) that for each $x \in C$,

$$-g(\tilde{u}(x))u(x) \geq a^*|(w + z)(x)|^2 - a^*|v(x)|^2 \geq \alpha|(w + z)(x)|^2 - a^*|v(x)|^2.$$

Then we obtain that for each $x \in A \cup C$,

$$
\begin{aligned}
&- \lambda_k|w(x)|^2 - \lambda_{k-1}|z(x)|^2 - g(\tilde{u}(x))u(x) \\
&\quad \geq -a^*|v(x)|^2 + (\alpha - \lambda_k)|w(x)|^2 + (\alpha - \lambda_{k-1})|z(x)|^2 + 2\alpha w(x)z(x).
\end{aligned} \tag{15}
$$

Let $x \in B$. Then we have from (4) that

$$-g(\tilde{u}(x))u(x) \geq b_*|(w + z)(x)|^2 - b_*|v(x)|^2. \tag{16}$$

Then it follows that

$$
\begin{aligned}
&- \lambda_k|w(x)|^2 - \lambda_{k-1}|z(x)|^2 - g(\tilde{u}(x))u(x) \\
&\quad \geq -b_*|v(x)|^2 - \lambda_k|w(x)|^2 - \lambda_{k-1}|z(x)|^2 + b_*|(w + z)(x)|^2 \\
&\quad \geq -b_*|v(x)|^2 + (\alpha - \lambda_k)|w(x)|^2 + (\alpha - \lambda_{k-1})|z(x)|^2 \\
&\qquad + (b_* - \alpha)|(w + z)(x)|^2 + 2\alpha w(x)z(x) \\
&\quad \geq -\lambda|v(x)|^2 + (\alpha - \lambda_k)|w(x)|^2 + (\alpha - \lambda_{k-1})|z(x)|^2 \\
&\qquad + \{(\lambda - b_*)|v(x)|^2 - (\alpha - b_*)|(w + z)(x)|^2\} + 2\alpha w(x)z(x).
\end{aligned} \tag{17}
$$

Then since $|(w+z)(x)| < |v(x)| + \delta$, we find that

$$\begin{aligned} &- \lambda_k|w(x)|^2 - \lambda_{k-1}|z(x)|^2 - g(\tilde{u}(x))u(x) \\ &\quad \geq -\lambda|v(x)|^2 + (\alpha - \lambda_k)|w(x)|^2 \\ &\qquad + (\alpha - \lambda_{k-1})|z(x)|^2 + M + 2\alpha w(x)z(x). \end{aligned} \tag{18}$$

Thus combining (15) and (18) with (14), we obtain

$$\begin{aligned} \langle Tu, u\rangle &\geq \|v\|^2 - \lambda|v|^2 + (\alpha - \lambda_k)|w|^2 + (\alpha - \lambda_{k-1})|z|^2 + M|\Omega| + 2\alpha\langle w, z\rangle \\ &\geq \omega\|v\|^2 + (\alpha - \lambda_k)|w|^2 + (\alpha - \lambda_{k-1})|z|^2 - M|\Omega|. \end{aligned}$$

From the inequality above, we can see that there exists $r > 0$ such that $\langle Tu, u\rangle \geq 0$ for all $u \in H_0^1$ with $\|u\| \geq r$. This proves the result.

Now we set $S = \{v \in H_1 + H_3 \colon \|v\| \leq r\}$, $S_1 = \{k\phi \colon s \leq k \leq r\}$, and $S_2 = \{k\phi \colon -r \leq k \leq -s\}$. Let $E = H_0^1$ and $K_i = S \times S_i$ $(i = 1, 2)$. We will show that the condition (P) holds with K replaced by K_i $(i = 1, 2)$. Suppose that $u \in \partial K_1$. Then $\|P_2u\| = s$ or $\|u\| \geq r$ holds. If $\|P_2u\| = s$, then we have by Lemma 2 that $\langle Tu, u - 2P_2u\rangle \geq 0$. Since $s < 2s < r$, we find that $2P_2u \in \operatorname{int} K_1$. If $\|u\| \geq r$ and $\|P_2u\| = d > s$, then by Lemma 3, we have that $\langle Tu, u - \varepsilon u\rangle = (1 - \varepsilon)\langle Tu, u\rangle \geq 0$ for each $\varepsilon > 0$ with $s/d < \varepsilon < 1$. Since $\varepsilon\|u\| < \|u\|$ and $\varepsilon\|P_2u\| > s$, we have $\varepsilon u \in \operatorname{int} K_1$. Thus we have shown that (P) is satisfied. Then by Theorem A, there exists $u_1 \in K_1$ such that $Tu_1 = 0$. Similarly, we have that there exists $u_2 \in K_2$ such that $Tu_2 = 0$. Since $K_1 \cap K_2 = \emptyset$, $u_1 \neq u_2$ and this completes the proof of the result.

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