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ON THE REPRESENTATION FORMULAS FOR THE FUNCTIONS IN CLASS $\Sigma^*(p, w_0)$

CHANG YUH LIN

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ABSTRACT. We prove the converse of J. Miller's integral representation theorem for functions in the class $\Sigma^*(p, w_0)$ of starlike meromorphic univalent functions. As an application we improve the bounds of the modulus. We also observe some properties of the class $\mathrm{ST}(P) = \bigcup_{w_0} \Sigma^*(p, w_0)$.

1. Introduction. Let $S(p)$ denote the set consisting of all functions $f(z) = z + a_2 z^2 + \cdots$ ($|z| < p$), which are univalent meromorphic in the unit disk $\mathbf{D} = \{z\colon |z| < 1\}$ except for a simple pole at p ($0 < p < 1$). J. Miller [**1**] introduced a subclass $\Sigma^*(p, w_0)$ of $S(p)$ which consists of functions $f(z)$ satisfying the condition

$$\text{(1)} \qquad \operatorname{Re}(Q_p(f, z, w_0)) = \operatorname{Re}\left(\frac{zf'(z)}{f(z) - w_0} + \frac{p}{z-p} - \frac{pz}{1-pz}\right) < 0 \qquad (z \in \mathbf{D}).$$

We know that if $f \in \Sigma^*(p, w_0)$, then the set $\mathbf{C} \setminus f(\mathbf{D})$ is starlike with respect to the point w_0. Miller [**1**, **2**] studied its properties and got many results. Among those he proved the theorem: If $f \in \Sigma^*(p, w_0)$, then

$$\text{(2)} \qquad \frac{f(z) - w_0}{w_0} = \frac{p}{(z-p)(1-pz)} \exp\left\{\frac{1}{\pi}\int_0^{2\pi} \log(1 - e^{it}z)\, dV(t)\right\}.$$

Here the function $V(t)$ is nondecreasing over $[0, 2\pi]$ and $\int_0^{2\pi} dV(t) = 2\pi$. In this paper, we prove that the converse is true. That is, if f has the representation (2), then $f \in \Sigma^*(p, w_0)$. Thus, we get the structural formula for $f \in \Sigma^*(p, w_0)$. As an application, the sharp bound of the modulus is obtained, which improves a theorem of Miller. We also determine the "starlike center region" and a dense subclass of $\mathrm{ST}(p) = \bigcup_{w_0} \Sigma^*(p, w_0)$.

2. Characteristic representation of $\Sigma^*(p, w_0)$.

THEOREM 1. *$f \in \Sigma^*(p, w_0)$ if and only if there is a probability measure $\mu(x)$ on $\partial\mathbf{D}$ so that*

$$\text{(3)} \qquad f(z) = w_0 + \frac{pw_0}{(z-p)(1-pz)} \exp\left(\int_{|x|=1} 2\log(1 - xz)\, d\mu(x)\right),$$

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where w_0 *and* μ *satisfy the equation*

$$w_0 = -\frac{1}{p + 1/p - 2\int_{|x|=1} x\, d\mu(x)}. \tag{4}$$

PROOF. We need only prove that (3) is in $\Sigma^*(p, w_0)$ for any probability measure μ on $\partial\mathbf{D}$. In fact, if μ is such a function then w_0 is uniquely determined by (4). The function defined by (3) is analytic in $\mathbf{D}$ except for a simple pole at p and $f(0) = f'(0) - 1 = 0$, and $f(z) \neq w_0$ for $z \in \mathbf{D}$. It remains to show f is univalent. We note that $g(z) = z\exp(-2\int_{|x|=1} \log(1 - xz)\, d\mu(x))$ is known to be a regular starlike function in S^*. Let $F(z) = pw_0/(f(z) - w_0)$, then

$$F(z) = \frac{(z-p)(1-pz)}{z} g(z) \tag{5}$$

is regular in $\mathbf{D}$. But from (5) we deduce $\operatorname{Re}(zF'/g) > 0$ for $z \in \mathbf{D}$ [**3**, Lemma 1]. Hence F is close-to-convex with respect to g. The close-to-convexity of F implies the univalence of f. This completes the proof.

For further studying the properties of the starlike meromorphic functions, we consider the family of all starlike functions in $S(p)$ and define

$$\mathrm{ST}(p) = \bigcup_{w_0} \Sigma^*(p, w_0).$$

COROLLARY. *The "starlike center region" of* $\mathrm{ST}(p)$ *is the closed disk*

$$\mathscr{D}_p = \left\{ w\colon \left| w + \frac{p(1+p^2)}{(1-p^2)^2} \right| \leq \frac{2p^2}{(1-p^2)^2} \right\}. \tag{6}$$

PROOF. Suppose $f \in \Sigma^*(p, w_0)$. Since $|\int_{|x|=1} x d\mu(x)| \leq 1$, we know $w_0 \in \mathscr{D}_p$ from (4). Let $\mathscr{P}$ denote the class consisting of functions $\gamma(z) = 1 + \gamma_1 z + \gamma_2 z^2 + \cdots$, which are regular with $\operatorname{Re}\gamma > 0$ in $\mathbf{D}$. The correspondence between $\mathscr{P}$ and probability measures $\{\mu\}$ given through $\gamma(z) = \int_{|x|=1} (1 + xz)/(1 - xz) d\mu(x)$ is one-to-one. The coefficient region $\{\gamma'(0) = 2\int_{|x|=1} x d\mu\colon \gamma \in \mathscr{P}\}$ is $\{w\colon |w| \leq 2\}$. Hence

$$\left\{ -\frac{1}{p + 1/p - \gamma'(0)} : \gamma \in \mathscr{P} \right\} = \mathscr{D}_p.$$

REMARK. $\mathscr{D}_p \subsetneqq \{w\colon p/(1+p)^2 \leq |w| \leq P/(1-p)^2\}$.

3. The compactness and a dense subclass of $\mathrm{ST}(p)$.

THEOREM 2. *The class* $\mathrm{ST}(p)$ *is compact and has a dense subclass consisting of the following functions*:

$$g(z) = w_0(g) + \frac{pw_0(g)}{(z-p)(1-pz)} \prod_{k=1}^{m} (1 - x_k z)^{\nu_k}, \tag{7}$$

where $\nu_k \geq 0$, $\Sigma_{k=1}^m \nu_k = 2$ $(m \in \mathbf{N})$, $|x_k| = 1$ *and*

$$w_0(g) = -\frac{1}{p + 1/p - \sum_{k=1}^m \nu_k x_k}.$$

PROOF. Since $\mathrm{ST}(p) \subset S(p)$, $\mathrm{ST}(p)$ is locally uniformly bounded in $\mathbf{D} \setminus \{p\}$. It remains to show that $\mathrm{ST}(p)$ is sequentially closed. Suppose $f_n \in \Sigma^*(p, w_0^{(n)})$ and

$f_n \to f$ locally uniformly in $\mathbf{D} \setminus \{p\}$. For $z \in D$, $\mathrm{Re}(Q_p(f_n, z, w_0^{(n)})) < 0$. There is a subsequence $w_0^{(n_k)}$ so that $w_0^{(n_k)} \to w_0 \in \mathscr{D}_p$. Therefore, for $z \in \mathbf{D}$ we have $\mathrm{Re}(Q_p(f, z, w_0)) \le 0$. According to the maximum principle for harmonic functions and the equation $Q_p(f, 0, w_0) = -1$, equality above cannot hold. Consequently, $f \in \Sigma^*(p, w_0) \subset \mathrm{ST}(p)$.

Now we show that for any $f \in \mathrm{ST}(p)$ there is a sequence $\{f_n\}$ such that each f_n can be represented in the form (7), so that $f_n \to f$ locally uniformly in $\mathbf{D} \setminus \{p\}$. If $f \in \mathrm{ST}(p)$ then it can be represented as in (3). For $\mu(x)$ there is a sequence of probability measures $\{\mu_n\}$ such that [**4**]

$$\int_{|x|=1} 2\log(1 - xz)\, d\mu_n \to \int_{|x|=1} 2\log(1 - xz)\, d\mu,$$

$$\int_{|x|=1} x d\mu_n \to \int_{|x|=1} x\, d\mu,$$

where $\mu_n(x) = \sum_{k=1}^m \lambda_k \delta_{x_k}$, $\lambda_k \ge 0$, $\sum_{k=1}^m \lambda_k = 1$, δ_k is point mass at x_k. Let

$$f_n(z) = w_0(f_n) + \frac{p w_0(f_n)}{(z-p)(1-pz)} \exp\left(\int_{|x|=1} 2\log(1 - xz)\, d\mu_n\right),$$

where

$$w_0(f_n) = -\frac{1}{p + 1/p - 2\int_{|x|=1} x d\mu_n}.$$

Then $f_n \to f$ for $z \in \mathbf{D} \setminus \{p\}$. The functions $\{f_n\}$ have the form (7). Thus, $f_n \to f$ locally uniformly by Vitali's theorem. This proves the theorem.

Having found a dense subclass of $\mathrm{ST}(p)$, which can be expressed in explicit form, it is possible to use this result to investigate the extremal problems of starlike meromorphic functions.

4. The bounds of the modulus.

THEOREM 3. *If $f \in \Sigma^*(p, w_0)$, then the distance between the star center point w_0 and the point z on the level curve $\{f(re^{i\theta}) \colon 0 \le \theta \le 2\pi\}$ must be restricted by the bounds*

$$\frac{p|w_0|(1-r)^2}{(r+p)(1+pr)} \le |f(z) - w_0| \le \frac{p|w_0|(1+r)^2}{|r-p|(1-pr)}, \tag{8}$$

and these bounds are sharp.

PROOF. It is sufficient to prove the result for the dense subclass. The following function is a member of the class S^*:

$$\frac{z p w_0}{(f(z) - w_0)(z-p)(1-pz)} = \frac{z}{\prod_{k=1}^m (1 - x_k z)^{\nu_k}}.$$

Using the growth theorem for the class S^* and the triangle inequality, we can derive (8). The function

$$\frac{pz}{(z-p)(1-pz)} = w_0 + \frac{p w_0}{(z-p)(1-pz)}(1+z)^2 \in \Sigma^*(p, w_0), \tag{9}$$

where $w_0 = -p/(1+p)^2$. It is obvious that the function (9) yields the equality in (8) for $z = \pm r$.

REMARK. The inequality on the right-hand side of (8) improves the corresponding bounds obtained by Miller

$$\left|\frac{f(z)-w_0}{w}\right| \leq \frac{2^{2-\alpha}p(1+r)^{\alpha}}{|r-p|(1-pr)}, \quad [1]$$

where $0 \leq \alpha \leq 2$.

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DEPARTMENT OF MATHEMATICS, NORTHWEST UNIVERSITY, XIAN SHAANXI, PEOPLE'S REPUBLIC OF CHINA