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FUNCTIONS WHOSE DERIVATIVE HAS POSITIVE REAL PART

R. R. LONDON

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ABSTRACT. In this paper we find a sharp upper bound for $|zf'(z)/f(z)|$, where f is a normalised analytic function with Re $f'(z) > 0$ in the unit disc.

Denote by R the class of functions f which are regular in $D = \{z : |z| < 1\}$, and satisfy $f(0) = 0$, $f'(0) = 1$, Re $f'(z) > 0$ $(z \in D)$. In a recent paper [**3**], D. K. Thomas proved, for some absolute constant K, that

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{K}{(1-r)\log(1/(1-r))} \qquad (0 < |z| = r < 1), \tag{1}$$

whenever $f \in R$. He also asked what the sharp bound for zf'/f might be, and it is this question that prompted the present paper. We prove

THEOREM. *Let $f \in R$. Then*

$$\frac{|f'(z)|}{\text{Re } f(z)/z} \le \frac{1+r}{(1-r)(-1-(2/r)\log(1-r))} \qquad (0 < |z| = r < 1),$$

with equality for all r in the case of the function $k \in R$ given by $k(z) = -z - 2\log(1-z)$.

Equality here, for $f = k$, occurs since

$$k'(r) = \frac{1+r}{1-r}, \quad \frac{k(r)}{r} = -1 - \frac{2}{r}\log(1-r) \qquad (0 < r < 1),$$

and it is clear that the upper bound in the theorem is also a sharp upper bound for $|zf'(z)/f(z)|$, whenever $f \in R$.

1. Proof of the theorem.

We shall use the following

LEMMA. *For ρ and t in* $[0,1]$,

$$-\tfrac{8}{3}t^3\rho + (11\rho - 1)t^2 + 4(1-4\rho)t + 1 + \tfrac{11}{3}\rho \ge 0.$$

PROOF. Denote the left side of the inequality by $g(\rho, t)$, and the square $[0,1] \times [0,1]$ by S. We have

$$g(\rho,0) = 1 + \tfrac{11}{3}\rho > 0, \qquad g(\rho,1) = 4(1-\rho) \ge 0 \qquad (0 \le \rho \le 1),$$
$$g(0,t) = -t^2 + 4t + 1 > 0, \qquad g(1,t) = \tfrac{2}{3}(1-t)^2(7-4t) \ge 0 \qquad (0 \le t \le 1),$$

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so that the minimum value of g on ∂S is zero. Next we consider the critical points (ρ, t) of g, given by the equations

$$8t^3 - 33t^2 + 48t - 11 = 0, \tag{2}$$

$$\rho(4t^2 - 11t + 8) + t - 2 = 0. \tag{3}$$

The cubic in (2) is increasing, and so has just one real root t, which satisfies $.279 \le t \le .28$. The function $(2-t)/(4t^2 - 11t + 8)$ is also increasing, and hence $.328 \le \rho \le .329$ when ρ is given by (3) and t by (2). By minimising each term of g over these values of ρ and t, we obtain $g(\rho, t) > 2$. Since the minimum value of g is attained on ∂S, or at a critical point of g inside S, the proof is now complete.

We can now prove the theorem. Let

$$k(z) = -z - 2\log(1-z),$$

then

$$k'(z) = \frac{1+z}{1-z},$$

which shows that $k \in R$. By putting $z = \rho e^{i\theta}$ and integrating with respect to ρ over $[0, r]$, we obtain

$$\frac{k(z)}{z} = \frac{1}{r}\int_0^r \frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}}\, d\rho \qquad (z = re^{i\theta}). \tag{4}$$

Now let $f \in R$, then f' has a Herglotz representation [**1**, p. 22]

$$f'(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z}\, d\mu(t), \tag{5}$$

which gives

$$\frac{f(z)}{z} = \frac{1}{2\pi}\int_0^{2\pi} \frac{k(ze^{-it})}{ze^{-it}}\, d\mu(t) \tag{6}$$

using (4), after a similar integration to the one above. Next let

$$\phi(r) = \max_{|z|=r} \left|\frac{1+z}{1-z}\right| \Big/ \operatorname{Re}\frac{k(z)}{z} \qquad (0 < r < 1), \tag{7}$$

and note that ϕ is well defined since $\operatorname{Re}(k(z)/z) > 0$ $(z \in D)$ by (4). Using (5) with (7), and then (6), we deduce

$$|f'(z)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1+e^{-it}z}{1-e^{-it}z}\right| d\mu(t) \le \frac{\phi(r)}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{k(ze^{-it})}{ze^{-it}}\, d\mu(t) = \phi(r)\operatorname{Re}\frac{f(z)}{z}.$$

Obviously, equality holds if $f = k$. To complete the proof we need to show that

$$\phi(r) = \frac{1+r}{1-r}\Big/\frac{k(r)}{r},$$

and this we shall do by verifying that the maximum in (7) is attained at $z = r$.

We have

$$\frac{\partial}{\partial\theta}\left(\left|\frac{1+z}{1-z}\right| \Big/ \operatorname{Re}\frac{k(z)}{z}\right)$$

$$= \left[\left(\operatorname{Re}\frac{k(z)}{z}\right)\left|\frac{1+z}{1-z}\right|\left(-\operatorname{Im}\frac{2z}{1-z^2}\right) \left|\frac{1+z}{1-z}\right| \frac{\partial}{\partial\theta}\left(\operatorname{Re}\frac{k(z)}{z}\right)\right] \Big/ \left(\operatorname{Re}\frac{k(z)}{z}\right)^2,$$

where $z = re^{i\theta}$. Now using (4), with the notation

$$J(r,x) = 1 - 2rx + r^2, \qquad x = \cos\theta,$$

we see that, apart from a nonnegative factor, the right side is

$$\left[-\left(\int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho\right)\left(\frac{r}{J(r,x)} + \frac{r}{J(r,-x)}\right) + 2\int_0^r \frac{\rho(1-\rho^2)}{J^2(\rho,x)}\,d\rho\right](\sin\theta).$$

So the maximum in (7) is attained at $z = r$ if the function in square brackets is nonpositive for $-1 \le x \le 1$, $0 < r < 1$. If we write the corresponding inequality as

$$\int_0^r \frac{d}{dt}\left[\int_0^t \frac{\rho(1-\rho^2)}{J^2(\rho,x)}\,d\rho - \frac{1}{2}\left(\int_0^t \frac{1-\rho^2}{J(\rho,x)}\,d\rho\right)\left(\frac{t}{J(t,x)} + \frac{t}{J(t,-x)}\right)\right] dt \le 0,$$

the integrand is

$$\frac{1}{2}(1-t^2)\left[\frac{t}{J^2(t,x)} - \frac{t}{J(t,x)J(t,-x)} - \left(\frac{1}{J^2(t,x)} + \frac{1}{J^2(t,-x)}\right)\int_0^t \frac{1-\rho^2}{J(\rho,x)}\,d\rho\right].$$

Thus it is sufficient to prove

$$\frac{1}{r}\int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho \ge \frac{J(r,-x)(J(r,-x) - J(r,x))}{J^2(r,x) + J^2(r,-x)} = \frac{4rxJ(r,-x)}{J^2(r,x) + J^2(r,-x)},$$

where $-1 \le x \le 1$, $0 < r < 1$. When x is nonpositive this is obvious, so we now assume $0 < x \le 1$, $0 < r < 1$.

The last inequality can be written as

$$\frac{1}{r}\int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho \ge \frac{1-u}{1+u^2},$$

where $u = J(r,x)/J(r,-x)$, and, since $0 < u < 1$, it is implied by

$$\frac{1}{r}\int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho \ge 1 - u = \frac{4rx}{J(r,-x)}.$$

Now for $0 \le \rho \le x$ we have $0 < J(\rho,x) \le 1 - \rho^2$, so that

$$\text{(8)} \qquad \frac{1}{r}\int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho \ge 1 \qquad (0 < r \le x),$$

and we have only to prove

$$\text{(9)} \qquad \int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho \ge \frac{4r^2x}{J(r,-x)} \qquad (0 < x \le r < 1).$$

Assume now that $0 < x \le r < 1$. We have

$$\frac{d}{dr}\left(\frac{1}{J(r,x)}\right) = \frac{-2(r-x)}{J^2(r,x)} \le 0,$$

and by using this with (8) we obtain

$$\begin{aligned}\int_0^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho &\ge x + \int_x^r \frac{1-\rho^2}{J(\rho,x)}\,d\rho \\ &\ge x + \frac{1}{J(r,x)}\int_x^r (1-\rho^2)\,d\rho \\ &= x + \frac{1}{J(r,x)}\left(r - \frac{1}{3}r^3 - x + \frac{1}{3}x^3\right).\end{aligned}$$

So a proof of (9) is reduced to that of

$$\frac{4r^2x}{J(r,-x)} \le x + \frac{1}{J(r,x)}\left(r - \frac{1}{3}r^3 - x + \frac{1}{3}x^3\right),$$

which we observe is true for $x = r$, since $4rx < J(r,-x)$. The inequality is equivalent to

$$(10)\quad \tfrac{1}{3}r^5 + \tfrac{11}{3}xr^4 - (8x^2 + \tfrac{2}{3})r^3 + (\tfrac{11}{3}x^3 + x)r^2 + (-1 + 2x^2 - \tfrac{2}{3}x^4)r - \tfrac{1}{3}x^3 \le 0,$$

and because it is true for $x = r$ it is implied by the inequality

$$-\tfrac{8}{3}x^3r + (11r^2 - 1)x^2 + (4r - 16r^3)x + r^2 + \tfrac{11}{3}r^4 \ge 0,$$

in which the left side is the derivative with respect to x of the left side of (10). If we now put $x = tr$, and then $\rho = r^2$, this inequality follows from the one in the lemma and the proof is complete.

2. Remarks. A general result of Ruscheweyh [**2**] has a direct bearing on the problem of maximising $|zf'/f|$ for functions f in R. It shows that the extreme function has the form

$$f'(z) = t\frac{1+\alpha z}{1-\alpha z} + (1-t)\frac{1+\beta z}{1-\beta z},$$

where $|\alpha| = |\beta| = 1$, $0 < t \le 1$. But in this approach the technical details seem rather more awkward than those given here.

Thomas [**3**] notes from (1) that for bounded functions f in R

$$(11)\qquad M(r,f') = O(1)\left[(1-r)\log\frac{1}{1-r}\right]^{-1},$$

where $M(r,f') = \max_{|z|=r}|f'(z)|$. The result of this paper shows that (11) also holds for $f \in R$, whenever $\mathrm{Re}(f(z)/z)$ is bounded.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY COLLEGE SWANSEA, SINGLETON PARK, SWANSEA, SA2 8PP WALES