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# ANALYTIC FUNCTIONS WITH RECTIFIABLE RADIAL IMAGES

MARVIN ORTEL AND WALTER SCHNEIDER

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ABSTRACT. We give a simple sufficient condition for an analytic function in the unit disk to have a radial image of finite length.

**1. Introduction.** If $f$ is an analytic function defined on the unit disk $D \equiv \{z \in \mathbf{C}\colon |z| < 1\}$, we define the *radial variation function of* $f$,

$$V(f,\cdot)\colon [0,2\pi) \to [0,\infty) \cup \{\infty\},$$

by the rule

$$V(f,\theta) \equiv \int_{[0,1)} |f'(re^{i\theta})|\, dr, \quad \text{all } \theta \in [0,2\pi).$$

In the present paper we prove that $V(f,\theta)$ is finite for at least one $\theta \in [0,2\pi)$ if $f$ is of moderate growth in the unit disk and $f'$ is bounded on some arc tending to the boundary of $D$.

THEOREM 1. *Suppose $f$ is analytic in $D$, $\mu \in [0,1)$ and*

$$\sup_{z\in D} |f(z)|(1-|z|)^{\mu} < \infty.$$

*In addition, suppose $\gamma\colon [0,1) \to D$ is continuous and one-to-one, $\lim_{t\uparrow 1} |\gamma(t)| = 1$, and*

$$\sup_{t\in[0,1)} |f'(\gamma(t))| < \infty. \tag{1}$$

*Then there exists $\theta \in [0,2\pi)$ such that $V(f,\theta) < \infty$.*

Theorem 1 is proved in §3, by means of the harmonic majorization discussed in §2. In §4 we present a corollary directed to an open question concerning the radial images of bounded functions in the disk.

**2. Majorization.** The following lemma shows how the *radial* growth of a function which is subharmonic in the upper half plane is *restricted* in the case that it is *bounded* on a curve terminating at the origin.

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LEMMA 1. *Suppose $h$, $\Gamma$, and $\varepsilon$ meet the following conditions:*

(1) *$h$ is subharmonic in $H^+ \equiv \{z \in \mathbf{C}\colon \operatorname{Im} z > 0\}$ and $\varepsilon \in (0,\infty)$.*

(2) *$h(x+iy) \leq \log \frac{1}{y}$ if $x+iy \in H^+$ and $x^2+y^2 \leq 1$.*

(3) *$\Gamma\colon [0,1) \to H^+$ is continuous and one-to-one, $\Gamma(0) = i$, $\lim_{t\uparrow 1}\Gamma(t) = 0$, and $|\Gamma(t)| \leq 1$ for all $t \in [0,1)$.*

(4) $\sup_{t\in[0,1)} h(\Gamma(t)) < \infty$.

*Then we have*

(5) $\sup_{y\in(0,1]}[h(iy) + (\frac{1}{2}+\varepsilon)\log y] < \infty$.

PROOF. Choose $M \in (0,\infty)$, $\alpha \in (\pi/2,\pi)$ so that $\pi/2\alpha < 1/2+\varepsilon$, $h(\Gamma(t)) \leq M$ for all $t \in [0,1)$, and $h(e^{i\theta}) \leq M$ for all $\theta \in [\pi-\alpha,\alpha]$. Finally, select $\delta \in (0,\infty)$. We shall prove

$$h(iy_0) \leq M + 2\delta + \frac{\pi}{2\alpha}\log\frac{1}{\sin\alpha} + \frac{\pi}{2\alpha}\log\frac{1}{y_0}, \tag{6}$$

all $y_0 \in (0,1]$.

Note that (6) implies (5), since $\pi/2\alpha < 1/2+\varepsilon$, and both $y_0$ and $\delta$ are arbitrary.

We now prove (6). Corresponding to fixed $y_0$ and $\delta$, choose numbers $\rho$ and $d$ such that

$$\begin{gathered} 0 < d < \rho < y_0, \quad \frac{2\rho y_0}{d^2+y_0^2} \leq \delta, \\ \frac{\rho}{y} > \log\frac{1}{y} \quad \text{for all } y \in (0,d]. \end{gathered} \tag{7}$$

Also, introduce the following sets which depend on the chosen number $d$:

$$\begin{aligned}
&\Gamma_1 \equiv \{de^{i\theta} : 0 \leq \theta \leq \pi\} \cup \{re^{i\alpha} : d \leq r \leq 1\} \cup \{e^{i\theta} : \pi/2 \leq \theta \leq \alpha\},\\
&\Gamma_2 \equiv \{de^{i\theta} : 0 \leq \theta \leq \pi\} \cup \{re^{i(\pi-\alpha)} : d \leq r \leq 1\} \cup \{e^{i\theta} : \pi-\alpha \leq \theta \leq \pi/2\},\\
&t^* \equiv \inf\{t \in [0,1) : |\Gamma(t)| = d\},\\
&\gamma^* \equiv \{\Gamma(t) : 0 \leq t \leq t^*\},\\
&\mathscr{L} \equiv \{z \in \mathbf{C} : \Gamma_1 \cup \gamma^* \text{ separates } z \text{ from } \infty\},\\
&\mathscr{R} \equiv \{z \in \mathbf{C} : \Gamma_2 \cup \gamma^* \text{ separates } z \text{ from } \infty\}.
\end{aligned}$$

We claim $iy_0 \in \mathscr{L} \cup \mathscr{R} \cup \gamma^*$. This follows from Janisewski's Theorem [**1**, p. 362], since the set

$$(\Gamma_1 \cup \gamma^*) \cap (\Gamma_2 \cup \gamma^*) = \{de^{i\theta} : 0 \leq \theta \leq \pi\} \cup \gamma^*$$

is connected and $\Gamma_1 \cup \Gamma_2$ separates $iy_0$ from $\infty$.

We may therefore assume $iy_0 \in \mathscr{L}$: indeed, (6) is immediate if $iy_0 \in \gamma^*$, and if $iy_0 \in \mathscr{R}$ we merely replace $h$ and $\Gamma$ by their reflections in the imaginary axis.

Assuming that $iy_0 \in \mathscr{L}$, let $\Omega$ denote the set of complex numbers which may be connected to $iy_0$ by a path not crossing $\Gamma_1 \cup \gamma^*$. By definition of $\Omega$, it follows that $\operatorname{Bdry}\Omega \subset \Gamma_1 \cup \gamma^*$ (here $\operatorname{Bdry}\Omega$ denotes the $\mathbf{C}$-boundary of $\Omega$). Moreover, if we set $m \equiv \min[d\cdot\sin(\alpha), \inf_{z\in\gamma^*}\operatorname{Im} z]$, we have

$$\overline{\Omega} \subset \mathbf{S} \equiv \{z \in \mathbf{C}\colon d \leq |z| \leq 1 \text{ and } \operatorname{Im} z \geq m\}.$$

Indeed, since $m > 0$, any point outside $S$ either lies on $\Gamma_1$ or may be connected to $\infty$ without crossing $\Gamma_1 \cup \gamma^*$. So, if $z \in \Omega$ and $z \notin S$, we could connect $iy_0$ to

$z$ and $z$ to $\infty$ without crossing $\Gamma_1 \cup \gamma^*$: but this contradicts the assumption that $iy_0 \in \mathscr{L}$. Therefore, $\overline{\Omega}$ is a compact subset of $H^+$, $\overline{\Omega} \subset S$, and

$$\operatorname{Bdry} \Omega \subset (\Gamma_1 \cup \gamma^*) - \{d, -d\}.$$

We may now complete the proof. For $z = x + iy \in H^+$ define

$$u(z) \equiv (M+\delta) + \left(\log \frac{1}{|z| \sin \alpha}\right)\left(\frac{\arg z}{\alpha}\right) + \rho\left[\frac{y}{(x+d)^2+y^2} + \frac{y}{(x-d)^2+y^2}\right].$$

Since $(\log|z|)(\arg z) = \frac{1}{2}\operatorname{Im}(\log z)^2$ in $H^+$, we see that $u$ is harmonic in $H^+$. Note that each term of $u$ is nonnegative on Bdry $\Omega$ since $|z| \le 1$ for all $z \in \overline{\Omega} \subset S$. Now, if $x^2 + y^2 = d^2$ and $y > 0$, we have

$$u(x+iy) > \rho\left[\frac{y}{(x+d)^2+y^2} + \frac{y}{(x-d)^2+y^2}\right] = \frac{\rho}{y} > \log\frac{1}{y} \ge h(x+iy),$$

by (7) and our hypotheses. If $x + iy = re^{i\alpha}$ and $d \le r \le 1$, then

$$u(x+iy) \ge \left(\log\frac{1}{r \sin \alpha}\right)\left(\frac{\alpha}{\alpha}\right) = \log\frac{1}{y} \ge h(x+iy).$$

If $z \in \gamma^*$ or $z = e^{i\theta}$ with $\pi/2 \le \theta \le \alpha$, then $u(z) > M \ge h(z)$. Since Bdry $\Omega \subset \Gamma_1 \cup \gamma^* - \{d, -d\}$, we have shown

$$h(z) - u(z) \le 0 \quad \text{for all } z \in \operatorname{Bdry} \Omega.$$

Since $h - u$ is subharmonic in $H^+$ and $\overline{\Omega}$ is a compact subset of $H^+$, we conclude

$$h(z) - u(z) \le 0 \quad \text{for all } z \in \Omega.$$

Since $iy_0 \in \Omega$ we have

$$h(iy_0) \le u(iy_0) = (M+\delta) + \left(\log\frac{1}{y_0} + \log\frac{1}{\sin\alpha}\right)\cdot\frac{\pi}{2\alpha} + \rho\left[\frac{2y_0}{d^2+y_0^2}\right].$$

Now (6) follows from the setup in (7).

**3. Proof of Theorem 1.** From the general hypotheses it follows that $\sup_{z \in D} |f'(z)|(1-|z|)^{\mu+1} < \infty$. Now consider hypothesis (1). By theorems of G. MacLane [**2**, Theorem 1 and Theorem 14 in G. Piranian's review], there is a dense set of points on the unit circle which are endpoints of asymptotic paths of $f'$ (possibly corresponding to infinite limits). This fact allows us to assume that $\lim_{t \uparrow 1} \gamma(t)$ exists. For otherwise there is a nonempty open interval $I \subset (0, 2\pi)$ such that for each $\theta \in I$ there is a corresponding sequence $(t_n)$ from $[0,1)$ with $\lim_{n\uparrow\infty} \gamma(t_n) = e^{i\theta}$. By the existence of asymptotic paths of $f'$, we may choose $\theta_0$ and $\theta_1 \in I$, and a Jordan arc $\sigma\colon (0,1) \to D$ such that

$$\lim_{t\downarrow 0} \sigma(t) = e^{i\theta_0}, \qquad \lim_{t\uparrow 1} \sigma(t) = e^{i\theta_1},$$
$$\lim_{t\downarrow 0} f'(\sigma(t)) = a, \qquad \lim_{t\uparrow 1} f'(\sigma(t)) = b,$$

where $a$ and $b$ are elements of the *extended* complex plane. By elementary topology we see that for each $\varepsilon \in (0,1)$ we have

$$\{\gamma(t) : 1-\varepsilon < t < 1\} \cap \{\sigma(t) : 0 < t < 1\} \neq \varnothing.$$

Since $\lim_{t\uparrow 1} |\gamma(t)| = 1$ and $\sup_{t\in[0,1)} |f'(\gamma(t))| < \infty$, we may conclude that both $a$ and $b$ are (finite) complex numbers. Hence, if $\lim_{t\uparrow 1} \gamma(t)$ does not exist we may replace $\gamma$ by the parametric arc

$$t \to \sigma\left(\frac{1+t}{2}\right), \qquad t \in [0,1).$$

We may therefore assume, without loss of generality, that $\lim_{t\uparrow 1} \gamma(t) = 1$, $\gamma(0) = 0$, and $\operatorname{Re}\gamma(t) \geq 0$, all $t \in [0,1)$.

Since $\sup_{z\in D} |f'(z)|(1-|z|)^{\mu+1} < \infty$ it is easy to show that

$$\sup_{\substack{z\in H^+\\ |z|\leq 1}} \left| f'\left(\frac{i-z}{i+z}\right) \cdot \frac{2i}{(i+z)^2}\right| (\operatorname{Im} z)^{\mu+1} < \infty.$$

Hence, without loss of generality (multiply $f$ by a positive constant), we assume

$$\left| f'\left(\frac{i-z}{i+z}\right) \cdot \frac{2i}{(i+z)^2}\right| < \left(\frac{1}{y}\right)^{\mu+1} \qquad \text{if } z \equiv x+iy \in H^+ \text{ and } |z| \leq 1.$$

Define now

$$\Gamma(t) \equiv i\frac{1-\gamma(t)}{1+\gamma(t)}, \qquad t \in [0,1),$$

$$h(z) \equiv \frac{1}{\mu+1} \log\left| f'\left(\frac{i-z}{i+z}\right) \cdot \frac{2i}{(i+z)^2}\right|, \qquad z \in H^+.$$

Then $h$, $\Gamma$ meet the conditions of Lemma 1 because of our simplifying assumptions. If $\varepsilon \in (0,\infty)$ we conclude

$$\sup_{y\in(0,1]} \left[\frac{1}{\mu+1} \log\left| f'\left(\frac{1-y}{1+y}\right) \cdot \frac{2}{(1+y)^2}\right| + (\tfrac{1}{2}+\varepsilon)\log y\right] < \infty$$

and hence that

$$\sup_{y\in(0,1]} \left| f'\left(\frac{1-y}{1+y}\right)\right| \frac{2y^{(1/2+\varepsilon)(1+\mu)}}{(1+y)^2} < \infty.$$

Since $\mu \in [0,1)$ we may choose $\varepsilon$ so that $(\frac{1}{2}+\varepsilon)(\mu+1) < 1$ and obtain

$$\int_{[0,1)} |f'(r)|\, dr = \int_{(0,1]} \left| f'\left(\frac{1-y}{1+y}\right)\right| \cdot \frac{2}{(1+y)^2}\, dy < \infty.$$

This establishes Theorem 1.

**4. Corollary for bounded functions.** Although the question of the rectifiability of radial images originates in Rudin's paper [**3**], it is still unknown whether *all* the radial images of a *bounded* analytic function $f$ can be of infinite length. However, by Corollary 1 (stated below), this is impossible if $f'$ has less than the maximal density of zeros possible for the derivative of a bounded analytic function in the disk.

To see this, define $A(f') \equiv \{a \in D\colon f'(a) = 0\}$ whenever $f$ is an analytic function in the disk. Then, if $f$ is bounded, we have

$$\sum_{a \in A(f')} (1 - |a|)^{1+\varepsilon} < \infty, \quad \text{for all } \varepsilon > 0$$

(see [**4**, pp. 204–205]). By Corollary 1, if $V(f,\theta) = \infty$ *for all* $\theta \in [0, 2\pi)$ it must actually be the case that $\sum_{a \in A(f')}(1 - |a|) = \infty$.

COROLLARY 1. *Suppose $f$ is analytic in $D$, $\sum_{a \in A(f')}(1 - |a|) < \infty$, $\mu \in [0, 1)$, and* $\sup_{z \in D} |f(z)|(1-|z|)^{\mu} < \infty$. *Then there exists $\theta \in [0, \pi)$ such that $V(f,\theta) < \infty$.*

PROOF. Let $B$ denote the Blaschke product with zero set $A$ and set $g \equiv f'/B$. Since $\log|g|$ is harmonic in $D$ and $D$ is simply connected, there is a simple parametric arc $\gamma\colon [0,1) \to D$ such that $\gamma(0) = 0$, $\lim_{t \uparrow 1} |\gamma(t)| = 1$, and $\log|g(\gamma(t))| = \log|g(0)|$ for all $t \in [0,1)$. Thus $|f'(\gamma(t))| \leq |g(0)|$ for all $t$. Now we apply Theorem 1.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HAWAII 96822

DEPARTMENT OF MATHEMATICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA K1S 5B6