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INVARIANT LAGRANGIAN SUBSPACES

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ABSTRACT. It is proved that on Hilbert spaces with strong symplectic form, every symplectic operator $I + C$ with C compact has an invariant Lagrangian subspace.

1. Introduction. It is a well-known fact that every symplectic operator on a finite-dimensional complex vector space has a Lagrangian invariant subspace. In this note we generalize the result on invariant Lagrangian subspaces to symplectic operators of the form $I + C$ with C compact on a Hilbert space with strong symplectic form. The proof uses a method due to Arveson and Feldman [**1**].

The relation between Lagrangian subspaces and complex structures is of importance in the quantization problem [**8**]. The result on existence of invariant Lagrangian subspaces proved here is a step towards extending the results of Paneitz [**4**] which assume a bound on C.

Let $(H, \langle\ ,\ \rangle)$ be a separable Hilbert space over the field $\mathbb{C}$ equipped with a strong symplectic form ω, i.e. a skew-symmetric continuous bilinear form such that the induced mapping $\hat{\omega}\colon H \to H^*$ defined by $\langle \hat{\omega}x, y\rangle = \omega(x, y)$ is an isomorphism.

Let $L(H)$ denote the space of continuous linear automorphisms of H. An element $A \in L(H)$ is said to be symplectic if $\omega(Ax, Ay) = \omega(x, y)$, $\forall x, y \in H$ and the subgroup of $\mathrm{GL}(H)$ consisting of symplectic operators with respect to ω is denoted by $\mathrm{Sp}(H, \omega)$. The subgroup of $\mathrm{Sp}(H, \omega)$ consisting of operators of the form $(I + C)$ with C compact is denoted by $\mathrm{Sp}_C(H, \omega)$ and the corresponding Lie algebra by $\mathbf{sp}_C(H, \omega)$.

Let $L \subset H$ be a linear subspace. Then we define the ω-orthogonal complement $L^\perp$ of L by

$$L^\perp = \{x \in H | \omega(x, y) = 0,\ \forall y \in L\}.$$

A subspace $L \subset H$ is said to be isotropic if $L \subset L^\perp$, coisotropic if $L^\perp \subset L$ and symplectic if $L \cap L^\perp = \{0\}$. A closed isotropic subspace L is said to be Lagrangian if there exists another closed isotropic subspace L' such that $H = L \oplus L'$.

The proof of [**7**, Proposition 5.1] extends easily to Hilbert spaces over $\mathbb{C}$ to show that a maximal isotropic subspace (i.e. $L = L^\perp$) is Lagrangian in the present case.

We are now ready to state our main theorem.

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THEOREM 1. *Let (H, ω) be as above. Then, for any $A \in \mathrm{Sp}_C(H, \omega)$, there exists a Lagrangian subspace $L \subset H$ which is invariant under A, i.e. $AL \subset L$. The same statement holds for $X \in \mathbf{sp}_C(H, \omega)$.*

Theorem 1 will be proved in §2. In the corollaries below we record some facts that can be proved using Theorem 1. It is also possible to prove these directly using finite-dimensional approximations of C and the continuity of spectral data.

Let C be a compact operator on H. Then the spectrum of C consists of an at most countable number of points with 0 as their only point of accumulation. The nonzero points in the spectrum are always eigenvalues of finite multiplicity, but the point 0 which is always in the spectrum can be "infinitely degenerate" and indeed, a compact operator may lack eigenvalues. See [**5**] for discussion and examples. To study the properties of the spectrum of symplectic operators in Sp_C we use the following normal form, which is a standard result on symplectic operators with an invariant Lagrangian subspace; see [**6**, Proposition 2.5] for the real case.

PROPOSITION 1. *Let (H, ω) be as above, let $A \in \mathrm{Sp}(H, \omega)$ and assume that A has an invariant Lagrangian subspace L. Then there is a symplectomorphism $\rho\colon (H, \omega) \to (L \oplus L^*, \Omega)$, where Ω is the canonical symplectic form on $L \oplus L^*$, which transforms A to the form*

$$\rho \circ A \circ \rho^{-1} = \begin{bmatrix} B & BE \\ 0 & (B^*)^{-1} \end{bmatrix},$$

where $B\colon L \to L$ is invertible and $E\colon L^* \to L$ defines a symmetric bilinear form on L^*, which in this case is bounded. For $X \in \mathbf{sp}(H, \omega)$ the corresponding form is

$$\rho \circ X \circ \rho^{-1} = \begin{bmatrix} Z & E \\ 0 & -Z^* \end{bmatrix},$$

where Z is any bounded operator and E is as above.

This normal form immediately gives the following characterization of the spectrum of symplectic operators in Sp_C.

COROLLARY 1. *Let (H, ω) be a symplectic Hilbert space over $\mathbf{R}$ with strong symplectic form ω and let $A \in \mathrm{Sp}_C(H, \omega)$. Let λ be in the spectrum of A. Then $\lambda, \overline{\lambda}, \lambda^{-1}\overline{\lambda}^{-1}$ all have the same multiplicity. In particular, the multiplicity of 1 is even or infinite.*

For $X \in \mathrm{sp}_C(H, \omega)$, the corresponding statement is that if λ is in the spectrum of X, then $\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda}$ all have the same multiplicity.

PROOF. Extend A and ω to the complex Hilbert space $H \otimes \mathbf{C}$. Then Theorem 1 gives an invariant Lagrangian subspace so we can apply Proposition 1. The result now follows from standard spectral theory. □

Let $\mathrm{Sp}_r(H, \omega)$ (with Lie-algebra $\mathbf{sp}_r(H, \omega)$) denote the space of symplectic operators of the form $A = I + C$ with $C \in L^r(H)$, where, for $r = 1$, L^1 denotes the space of traceclass operators on H and for $r = 0$, L^0 denotes the space of operators on H with finite rank.

For an operator of the form $A = I + C$ with $C \in L^1(H)$,

$$\det(A) = \prod_{i=1}^{N(A)} \lambda_i,$$

where $\det(A)$ denotes the Fredholm determinant of A, $\lambda_i(A)$ are its eigenvalues, with multiplicity $N(A)$ (see [**5**]). Similarly, for $X \in L^1(H)$,

$$\operatorname{tr}(X) = \sum_{i=1}^{N(X)} \lambda_i(X).$$

Applying this to the result in Corollary 1 gives

COROLLARY 2. *Let* $A \in \mathrm{Sp}_1(H,\omega)$. *Then* $\det(A) = 1$. *Let* $X \in \mathbf{sp}_1(H,\omega)$. *Then* $\operatorname{tr}(X) = 0$.

2. Proof of Theorem 1. We will need a few preliminaries.

LEMMA 1. *Let* $A \in \mathrm{Sp}_C(H,\omega)$ *and let* $L \subset H$ *be a closed invariant subspace of* A. *Then* $L^\perp$ *is a closed invariant subspace of* A.

PROOF. Since $A \in \mathrm{Sp}_C$, $A|_L \colon L \to L$ is injective and has index zero and hence is surjective. Thus, every $y \in L$ satisfies $y = Ax$ for some $x \in L$ which implies $A^{-1}y = A^{-1}Ax = x \in L$. This shows that $A^{-1}L = L$. Let now $x \in L^\perp$ and note that $\omega(Ax,y) = \omega(x,A^{-1}y) = 0$ for all $y \in L$. It follows by the definition of $L^\perp$ that $Ax \in L^\perp$. □

LEMMA 2. *Let* $L \subset H$ *be an isotropic subspace. Then* $L^\perp/L$ *has an induced symplectic form* ω'. *Let* $W \subset L^\perp/L$ *be an isotropic subspace with respect to* ω'. *Then there is a unique subspace* W_0 *of* $L^\perp$ *such that* $W_0/L = W$. *Further,* W_0 *is isotropic.*

PROOF. This is standard, except for the last statement. Let W_0 be the (unique) preimage of W under the projection $\pi\colon L^\perp \to L^\perp/L$. Then, for $x, y \in W^0$ we have $\omega(x,y) = \omega'(\pi(x),\pi(y)) = 0$. □

LEMMA 3. *Let* $A \in \mathrm{Sp}_0(H,\omega)$. *Then* A *has an invariant Lagrangian subspace.*

REMARK. The proof of Lemma 3 is a straightforward extension of the proof for the finite-dimensional case (see [**2**]) but is included since it is used as a model for the final proof.

PROOF. We can use the same method as in the finite-dimensional case, since $A \in \mathrm{Sp}_0(H,\omega)$ implies that A has a complete set of generalized eigenvectors. Thus, let $e \in H$ be an eigenvector of A and let L denote the span of e. Then L is an invariant isotropic subspace for A.

Consider the set of all nontrivial invariant isotropic subspaces for A. By Zorn's Lemma, this has a maximal element which we denote by L. We claim that L is also maximal isotropic in H and hence Lagrangian by [**7**, Proposition 5.1].

To prove this, we assume the contrary, i.e. that L is not maximal isotropic in H. In particular, this means that $L^\perp/L$ is nontrivial. Note that $L^\perp$ is also invariant by Lemma 1 so we get an induced operator $A_{L^\perp/L}$ on $L^\perp/L$. Clearly, $A_{L^\perp/L} \in \mathrm{Sp}_0(L^\perp/L, \omega_{L^\perp/L})$ so we again find an eigenvector which spans a one dimensional isotropic subspace $W \subset L^\perp/L$. Thus, we can apply Lemma 2 to get an invariant isotropic subspace $L' \subset H$ such that $L \subset L'$ with strict inclusion. Hence, L is not maximal among the invariant isotropic subspaces for A, which is absurd.

We have derived a contradiction from the assumption that a subspace which is maximal among the A-invariant isotropic subspaces is not maximally isotropic in H. This completes the proof of Lemma 3. □

Let us recall some standard material concerning Hilbert spaces with a strong symplectic form. Let (H,ω) be as above and let a Lagrangian subspace $W \subset H$ be given. Then (see [**2**]) there is a symplectomorphism ρ between (H,ω) and (V,σ), where $V = W \times W^*$ and σ is the canonical symplectic structure on V, defined as follows: Let $(x_1, x_2), (y_1, y_2) \in V = W \times W^*$. Then we set

$$\sigma\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_2 \\ y_2 \end{pmatrix}\right) = (x_1, y_2) - (x_2, y_1), \tag{2.1}$$

where the pairings are the obvious ones.

Clearly, given (H,ω) we can, by choosing a Lagrangian subspace and using the map $\rho\colon (H,\omega) \to (V,\sigma)$ to form the isomorphism $A \to \rho^{-1} \circ A \circ \rho\colon L(H) \to L(V)$ without loss of generality assume that $H = W \times W^*$ and that ω is the canonical symplectic structure.

Now define a linear mapping $\hat{\omega}\colon H \to H^*$ by $\omega(x,y) = \langle \hat{\omega}x, y\rangle_{H^*,H}$ for $x, y \in H$. In matrix form, $\hat{\omega}$ can be represented as

$$\hat{\omega}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \in W^* \times W = H^*,$$

where the I's are the identity operators of W and W^*. Further, define its adjoint $\hat{\omega}^*$ by $\langle \hat{\omega}x, y\rangle_{H^*,H^*} = \langle x, \hat{\omega}^* y\rangle_{H,H}$ for $x, y \in H \times H^*$. Using (2.1) it is straightforward to verify that $\hat{\omega}^*$ in matrix form is represented by (let $(x_1, x_2) \in H^* = W^* \times W$)

$$\hat{\omega}^*\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \in W^* \times W = H^*.$$

From this it is immediate that $\hat{\omega}$ is an isometry and that

$$(\hat{\omega}^*)^{-1} = (\hat{\omega}^{-1})^* = \hat{\omega}, \tag{2.2}$$

where we have defined $(\hat{\omega}^{-1})^*$ analogously to $\hat{\omega}^*$. It is these special properties that make the following argument work.

Let $\phi\colon L(H) \to L(H)$ be defined by $\phi(X) = \hat{\omega}^{-1} X^* \hat{\omega}$ where $X^* \in L(H^*)$ is the conjugate operator to X, defined by $\langle Xy, z\rangle_{H,H^*} = \langle y, X^* z\rangle_{H,H^*}$ for $(y,z) \in H \times H^*$.

LEMMA 4. *Let $H = W \times W^*$ and let ω be the canonical symplectic form on H. Let $P\colon H \to H$ be an orthogonal projection onto a Lagrangian subspace of H. Then the following statements are true.*

(1) *$\phi(P)$ is an orthogonal projection onto a Lagrangian subspace,*
(2) *$\phi(P)P = P\phi(P) = 0$,*
(3) *$P + \phi(P) = I_H$, the identity operator on H.*

PROOF. Using the assumption that P is an orthogonal projection onto a Lagrangian subspace L of H, we can again find a symplectomorphism $\rho\colon H \to V = L \times L^*$. Due to the fact that $H = W \times W^*$ with the canonical symplectic structure ω we can take ρ to be an isometry. Denote the canonical symplectic structure of V by σ. Now we use ρ to move our operators to V:

$$X \to \rho_* X = \rho \circ X \circ \rho^{-1}\colon L(H) \to L(V),$$
$$X^* \to \rho_* X^* = \rho^{-*} X^* \rho^*\colon L(H^*) \to L(V^*).$$

Consider the effect of these operations on P and $\phi(P)$. By the construction of ρ, it takes $L\times H/L \to L\times\hat{\omega}(H/L) = L\times L^*$, so we see that $\rho_* P$ becomes projection onto the first component and $\rho_*\phi(P)$ becomes projection onto the second component.

By the linearity of ρ_* and the isometry properties of ρ, we can easily deduce the statements in the Lemma. For part (1), note that $\phi(P)$ is an orthogonal projection since $\rho_*\phi(P)$ is one and its range is $\rho^{-1}(0\times L^*)$ which is the image of a Lagrangian subspace by a symplectomorphism and hence Lagrangian.

Part (2) follows from the selfadjointness of P and $\phi(P)$ and the fact that $0 = (\rho_* P)(\rho_*\phi(P)) = \rho_*(P\phi(P))$.

The statement in part (3) follows from the fact that $\rho_*(P+\phi(P)) = I_V$, the identity operator on V. □

LEMMA 5. *Let (H,ω) and W be as in Lemma 4. Let $B \in L^1(W)$ be injective and nonnegative and assume that* $\operatorname{tr}(B) = 1/2$. *Let the state $\Omega\colon L(H) \to \mathbf{C}$ be given by $\Omega(X) = \operatorname{tr}((B\oplus B^*)\circ X)$. Then Ω is a faithful normal state on $L(H)$ and $\Omega(P) = 1/2$ for any orthogonal projection P onto a Lagrangian subspace of (H,ω).*

PROOF. The only statement that is not obvious is that $\Omega(P) = 1/2$. To see this, note that by Lemma 4,

$$1 = \Omega(I) = \Omega(P) + \Omega(\phi(P)).$$

Now, using the matrix form of $\hat{\omega}$ and $\hat{\omega}^{-1} = \hat{\omega}^*$, we get

$$\phi(B\oplus B^*) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} B^* & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B^* \end{bmatrix} = B\oplus B^*.$$

Thus we have $\Omega(\phi(P)) = \operatorname{tr}\phi((B\oplus B^*)\circ P)$. By the isometry properties of $\hat{\omega}$ we find that $\Omega(\phi(P)) = \Omega(P)$, so we get $2\Omega(P) = 1$. □

We now apply the above and the method of proof used in **[1]** to the problem of finding an invariant isotropic subspace for operators in $\mathrm{Sp}_C(H,\omega)$. This will allow us to generalize the method used in the proof of Lemma 3 to operators in Sp_C. By the remark before Lemma 4 we can, without loss of generality assume that $H = W\times W^*$ and that ω is the canonical symplectic form on H.

LEMMA 6. *Let (H,ω) be as above and let $A \in \mathrm{Sp}_C(H,\omega)$. Then A has a nontrivial invariant isotropic subspace.*

PROOF. If A has an eigenvector e, then the span of e is an invariant isotropic subspace. Thus, in the following we may without loss of generality assume that A has no eigenvectors.

It is well known that $\mathrm{Sp}_C(H,\omega)$ is the closure with respect to the norm topology of $\mathrm{Sp}_0(H,\omega)$ (to see this, use the surjectivity of the exponential mapping and apply the approximation property for compact operators in $\mathbf{sp}_C(H,\omega)$, cf. **[3]**). Thus, let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathrm{Sp}_0(H,\omega)$ such that $\|A_n - A\|_{L(H)} \to 0$ as $n\to\infty$. Let L_n be an invariant Lagrangian subspace for A_n, given by Lemma 3 and let P_n be the orthogonal projection onto L_n. Let Ω be as in Lemma 5, so that $\Omega(P_n) \equiv 1/2$.

Now let $P \in L(H)$ be an accumulation point of $\{P_n\}$ with respect to the weak topology and in the following, let $\{P_n\}$ denote a convergent subsequence. Then $\Omega(P) = 1/2$, so $P \neq 0$ and if we let $M = \{x | Px = x\}$, then $M \neq H$.

By construction, $(P_n - I)A_nP_n = 0$ for all n, so

$$\|(P_n - I)AP_n\| \leq \|(P_n - I)(A - A_n)P_n\| + \|(P_n - I)A_nP_n\| \leq \|A - A_n\|, \tag{2.2}$$

which tends to 0 as $n \to \infty$. This is the crucial property used in [**1**]. Following the proof in [**1**] we find that for $x \in M$, $P_nx \to x$ in norm, since

$$\|P_nx - x\|^2 = (P_nx, P_nx) - 2(P_nx, x) + (x, x) = -(P_nx, x) + (x, x) \to 0, \tag{2.3}$$

where we have used the projection property and selfadjointness of P_n It follows that

$$\omega(x, y) = \lim_{n\to\infty} \omega(P_nx, P_ny) = 0, \quad \forall x, y \in M,$$

so M is an isotropic subspace of H. To see that M is also invariant, note that for $x \in M$, $AP_nx \to Ax$ in norm by (2.3), so $P_nAP_nx \to PAx$ weakly. Now, using (2.2) we can conclude that $AP_nx \to PAx$ weakly, but by the above, $AP_nx \to Ax$ in norm, so $Ax = PAx$ and hence $Ax \in M$.

Finally, we have to check that M is nontrivial. Let $C = A - I$. Then, since $A \in \mathrm{Sp}_C(H, \omega)$ by assumption, C is compact. Let $x \in H$ be arbitrary. By the weak convergence of $P_n \to P$ and the compactness of C it follows that $CP_nx \to CPx$ in norm. Again by the weak convergence of P_n to P, we have that $P_nCP_nx \to PCPx$ weakly, but using (2.2) we find that $\|P_nCP_n - CP_n\| \to 0$, so in fact $CP_nx \to PCPx$ weakly. But we just proved that the left side converges in norm to CPx and it follows that $CPx = PCPx$.

This completes the proof since by the remark at the beginning, we can assume that A has no eigenvalues and in particular that the nullspace of C is trivial. Hence by choosing $x \in H$ so that $Px \neq 0$, which is always possible since $P \neq 0$, CPx provides a nonzero element of M. Thus, M is a nontrivial isotropic subspace, invariant under A. □

REMARK. Apart from using the projections P_n onto Lagrangian subspaces, the special state Ω and noting that M is isotropic, the proof of Lemma 6 follows very closely that of [**1**, Theorem, p. 62]. The author is grateful to Professor Arveson for pointing this paper out to him. □

We are now ready to complete the proof of Theorem 1. Let $A \in \mathrm{Sp}_C(H, \omega)$. By Lemma 6, there is a nontrivial invariant isotropic subspace for A. Consider as in the proof of Lemma 3, the set of all invariant isotropic subspaces for A. By Zorn's Lemma, this set has a maximal element which we call L. We wish to show that L is a maximal isotropic subspace of H and hence Lagrangian.

To do this, assume that L is not maximally isotropic in H. Then $L^\perp/L$ is a nontrivial symplectic space and arguing as in the proof of Lemma 3, we see that the induced operator $A_{L^\perp/L}$ is in $\mathrm{Sp}_C(L^\perp/L, \omega_{L^\perp/L})$ and hence by Lemma 6 that there is a nontrivial invariant isotropic subspace for $A_{L^\perp/L}$. By lifting to H as in the proof of Lemma 3, we find that L is not maximal among the invariant isotropic subspaces for A, a contradiction. It follows that L is maximally isotropic in H. The same method gives the result for $X \in \mathbf{sp}_C(H, \omega)$. This completes the proof of Theorem 1.

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