PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 4, August 1988

AN UPPER BOUND FOR THE PROJECTION CONSTANT

D. R. LEWIS

(Communicated by William J. Davis)

ABSTRACT. There is a positive function $\delta(n)$ of exponential order such that, for any normed space E of dimension $n \geq 2$, the projection constant of E satisfies $\lambda(E) \leq n^{1/2}[1 - \delta(n)]$.

The *projection constant* $\lambda(E)$ of a normed space E is the smallest $\lambda > 0$ with the following property; whenever X is a normed space with $E \subset X$ isometrically, there is a projection $u: X \to E$ with $\|u\| \leq \lambda$.

Kadec′ and Snobar [**2**] first proved that $\lambda(E) \leq n^{1/2}$ for every n-dimensional space E. In [**4**] the strict inequality $\lambda(E) < n^{1/2}$ was verified in case $n \geq 2$, but the infinite dimensional proof used there produced no concrete estimates for $\lambda(E)$. Here we establish an upper estimate for $\lambda(E)$ and an associated lower estimate for $\pi_1(E)$ as well.

The *absolutely summing constant* of E, $\pi_1(E)$, is the smallest $c > 0$ such that

$$\sum_{i \leq m} \|x_i\| \leq c \max_{|\varepsilon_i|=1} \left\| \sum_{i \leq m} \varepsilon_i x_i \right\|$$

for all choices of $x_1, x_2, \ldots, x_m \in E$.

THEOREM 1. *Let E be an n-dimensional space with $n \geq 2$. Then*
(a) $\pi_1(E) \geq n^{1/2} + (1/5)^{n+4}$, *and*
(b) $\lambda(E) \leq n^{1/2}[1 - n^{-2}(1/5)^{2n+11}]$.

Part (b) gives an exponential lower estimate for

$$\delta_n = \inf\{1 - n^{-1/2}\lambda(E) : \dim E = n\},$$

but it is possible that there is actually a power type lower estimate, i.e., that $\delta_n \geq cn^{-p}$ for some positive c and p. Complex examples constructed by H. König [**3**] show that $\delta_n \leq n^{-1/2}$ for infinitely many n.

Theorem 1 follows directly from an inequality relating the uniform and L_1-norms (Theorem 2) which will be proven first.

For the most part the notation and terminology is standard. Below T always denotes a compact Hausdorff space and $C(T)$ the space of real-valued continuous functions on T under the uniform norm $\|g\|_u$. For μ a regular Borel measure on T, $L_p(\mu)$ is the usual space of equivalence classes of functions for which $\|f\|_p = (\int |f|^p \, d\mu)^{1/p} < \infty$.

Received by the editors June 5, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46B20; Secondary 47A30.
Research partially supported by NSF grant DMS-8501117.

THEOREM 2. *Let $E \subset C(T)$ be a subspace of finite dimension $n \geq 2$. If μ is a Borel probability on T and c a constant with $\|g\|_u \leq c\|g\|_1$ for all g in E, then $c \geq n^{1/2} + (1/5)^{n+4}$.*

PROOF OF THEOREM 2. Write E_1 for E considered as a subspace of $L_1(\mu)$. Applying Theorem 1 of [**5**] to E_1, there is a basis $f_1, f_2, \ldots, f_n \in E$ so that, if $f = (\sum_{i\leq n} f_i^2)^{1/2}$, then

$$n\int f_i f_k f^{-1}\,d\mu = \delta_{ik}, \qquad 1 \leq i,\ k \leq n. \tag{1}$$

It follows that

$$\int f\,d\mu = \sum_{i\leq n}\int f_i^2 f^{-1}\,d\mu = 1. \tag{2}$$

We also need the uniform bound

$$f(s) \leq c, \quad \text{all } s \in T. \tag{3}$$

To see this, fix $s \in T$. The theorem's hypothesis gives

$$f(s)^2 \leq \left\|\sum_{i\leq n} f_i(s) f_i\right\|_u \leq c\int\left|\sum_{i\leq n} f_i(s) f_i\right|\,d\mu. \tag{4}$$

The Cauchy-Schwartz inequality shows that $|\sum_{i\leq n} f_i(s) f_i| \leq f(s) f$ pointwise on T, hence

$$f(s)^2 \leq c f(s)\int f\,d\mu = c f(s).$$

Now let $d\nu = f\,d\mu$, so ν is a probability on T. Below $\|\ \|_p$ will denote the norm in $L_p(\nu)$. For $1 \leq i \leq n$ let $h_i = n^{1/2} f_i f^{-1}$. It follows from (1) that $h_1, \ldots, h_n$ is orthonormal in $L_2(\nu)$ and clearly $\sum_{i\leq n} h_i(s)^2 = n$ everywhere on T. We first claim that the kernel

$$h(s,t) = \sum_{i\leq n} h_i(s) h_i(t)$$

satisfies

$$\iint |n^{-1/2}|h| - 1|^2\,d\nu\,d\nu \leq 2[c - n^{1/2}]. \tag{5}$$

To establish this first use (4); for $s \in T$

$$\begin{aligned} f(s) &\leq c\int\left|\sum_{i\leq n} f_i(s) f(s)^{-1} f_i f^{-1}\right| f\,d\mu \\ &= c n^{-1}\|h(s,\cdot)\|_1. \end{aligned}$$

From orthogonality of the h_i's

$$\|h(s,\cdot)\|_2 = \left(\sum_{i\leq n} h_i(s)^2\right)^{1/2} = n^{1/2},$$

and from the last two estimates

$$\|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 = n^{-1}\|h(s,\cdot)\|_2^2 - 2n^{-1/2}\|h(s,\cdot)\|_1 + 1$$
$$\leq 2[1 - n^{1/2}c^{-1}f(s)].$$

Integrating the last inequality, and taking (2) and (3) into account,

$$\int \|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 \nu(ds) \leq c\int \|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 \mu(ds)$$
$$\leq 2c[1 - n^{1/2}c^{-1}\int f(s)\mu(ds)] = 2[c - n^{1/2}].$$

Note that (5) implies $c \geq n^{1/2}$. Let M be the set of $s \in T$ for which

$$\|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 \leq 4[c - n^{1/2}]. \tag{6}$$

In case $c > n^{1/2}$, Chebyshev's inequality shows $\nu(M^c) \leq 1/2$, so that $\nu(M) \geq 1/2$ for any possible c. We now claim there are points $s_1, s_2, \ldots, s_N \in M$, with $N \leq 5^n$ and with the property that the sets

$$A_k = \{t \in T : h(s_k, t) > 7n/8\}, \qquad 1 \leq k \leq N,$$

cover M. To produce such points we use an ε-separation argument as follows.

For $s \in M$, $x(s) = (n^{-1/2}h_i(s))_{i\leq n}$ is on the unit sphere of l_2^n. Choose a set $B \subset M$ which is maximal with respect to the property that $\|x(s) - x(t)\|_{l_2^n} \geq 1/2$ for distinct $s, t \in B$. By [**1**, Lemma 2.4], $N = \operatorname{card} B \leq 5^n$. Writing $B = \{s_1, s_2, \ldots, s_N\}$, if $s \in M$ then for some $k \leq N$,

$$1/4 > \|x(s) - x(s_k)\|_{l_2^n}^2 = 2 - (2/n)h(s_k, s),$$

which means $s \in A_k$.

Finally note that $n \geq 2$ and $t \in A_k$ implies $|n^{-1/2}|h(s_k,t)| - 1|^2 \geq 1/20$. Using this and (6),

$$4[c - n^{1/2}]5^n \geq \sum_{k\leq N} \int_{A_k} |n^{-1/2}|h(s_k,\cdot)| - 1|^2 \, d\nu$$
$$\geq 1/20 \sum_{k\leq N} \nu(A_k) \geq 1/40,$$

from which $c \geq n^{1/2} + (1/5)^{n+4}$ follows. □

PROOF OF THEOREM 1. Let $T \subset E'$ be the closed unit ball under the norm topology. By Pietsch's integral representation theorem [**6**, Theorem 17.3.2] there is a Borel probability μ on T with

$$\|x\| \leq \pi_1(E)\mu(|\langle x, \cdot\rangle|).$$

Part (a) follows by applying Theorem 1 to the subspace $\{\langle x, \cdot\rangle|T : x \in E\} \subset C(T)$. Part (b) follows from part (a) and the following rephrasing of Theorem 2 of [**4**]: if $0 < \delta < 2$ and $\pi_1(E) \geq n^{1/2}(1 + \delta)$, then

$$\lambda(E) \leq n^{1/2}[1 - (2n)^{-1}(\delta/4)^2]. \quad \square$$

REFERENCES

1. T. Figiel, J. Lindenstrauss and V. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. **139** (1977), 53-94.
2. M. Ĭ. Kadec′ and M. G. Snobar, *Certain functionals on the Minkowski compactum*, Mat. Zametki **10** (1971), 453–457= Math. Notes **10** (1971), 694–696.
3. H. König, *Spaces with large projection constant*, Israel J. Math. **50** (1985), 181–188.
4. H. König and D. R. Lewis, *A strict inequality for projection constants*, J. Funct. Anal. **73** (1987), 328–332.
5. D. R. Lewis, *Finite dimensional subspaces of L_p*, Studia Math. **63** (1978), 207–212.
6. A. Pietsch, *Operator ideals*, North-Holland, Amsterdam, 1980.

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843