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APPROXIMATING PA

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ABSTRACT. It is shown that the ring of continuous real-valued functions on a compact Hausdorff space of dimension ≤ 1 is a BCS-ring. This result follows from a study of the preservation of the PA and BCS properties under certain ring-theoretic constructions.

Let R be a commutative ring with identity. Then R is said to be a PA-ring if each reachable system over R has its poles arbitrarily assignable, and is said to be a BCS-ring, if each basic submodule of a finitely generated projective R-module M contains a rank one projective summand of M (see below for undefined terms and a more detailed discussion of these ideas). Several recent papers have been concerned with PA- and BCS-rings: e.g. [**1**, **2**, **4**, **6**, **7**]. Among the more important results in this area is the following of Hautus and Sontag.

THEOREM 0. *Let X be a topological manifold with $C(X)$ the ring of continuous real-valued functions on X. Then $C(X)$ is a* PA*-ring if and only if X has dimension* ≤ 1.

There is an intriguing formal resemblance between this theorem and the purely algebraic result that a noetherian ring is a BCS-ring (and therefore a PA-ring) provided that it has dimension ≤ 1. (cf. [**6**, **7**]). Given the information that finitely generated projective modules over $C(X)$, where X is a finite simplicial complex, can be represented by projective modules over a noetherian ring of dimension $\leq \dim(X)$ (cf. [**5**]), and that projective modules play a key role in the study of PA- and BCS-rings, one might suspect that the relationship between the above results is more than one of formal resemblance. By adapting ideas of Swan [**5**] we are able to show in this paper that the Hautus-Sontag theorem is indeed a consequence of the algebraic result. Moreover, from our new vantage point it is easy to see how to obtain a generalisation of theorem 0 to the situation where X is a compact Hausdorff space of dimension ≤ 1.

This paper can also be viewed as a study of the preservation of the PA and BCS properties under certain ring theoretic constructions. One of the essential failures of PA-rings and BCS-rings is that they do not often respect ring-theoretic constructions. In Theorems 1 and 2 below, we wrest some reasonable behaviour from these recalcitrant objects.

Before we state and prove our main results we give the following definitions. A linear system over R is a pair of maps $f\colon U \to U$, $g\colon V \to U$ between finitely

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generated projective R-modules. We say that (f, g) is reachable if

$$\bigoplus_n V \xrightarrow{\oplus_n g} \bigoplus_n U \xrightarrow{(f)_i} U$$

is surjective, where $n = \operatorname{rank}(U)$, and pole assignable, if, given $(a_1, \ldots, a_n) \in R^n$, there is a map $k\colon U \to V$ such that $a_1, \ldots, a_n$ are the eigenvalues of $f + gk$. R is PA if every reachable system over R is pole assignable. R is PAF if every system with U and V free is pole assignable.

It can be shown that R is PA if and only if for every reachable system (f, g) over R the following condition holds: $\operatorname{Im}(g)$ contains a rank one projective summand of $\operatorname{Dmn}(f)$.

A ring R is a BCS-ring if every basic submodule of a finitely generated projective R-module P contains a rank one summand of P. It is fairly simple to show that R is a BCS-ring if and only if the following condition is satisfied: if G is an R-matrix having unit content then there is a matrix V such that GV is a $*$-matrix (i.e. GV has unit content and all 2×2 minors of GV are zero) (cf. [**1**, **6**]).

In the sequel $f\colon A \to B$ is a morphism of rings such that:

(1) B is a topological ring, $B*$ (the units of B) is open in B and $u \to u^{-1}$ is continuous on $B*$,

(2) $f(A)$ is dense in B,

(3) if $f(a)$ is sufficiently near 1 in B, then $a \in A*$.

Swan (5) also introduces the technical condition (SBI). We shall refrain from explicating this condition—it ensures that each finitely generated projective B-module is extended from A.

There are two theorems.

THEOREM 1. *Let $f\colon A \to B$ satisfy* (1)–(3). *If A is a* BCS-*ring, then B is a* BCS-*ring. If, in addition, f also satisfies* (*SBI*), *then B is a* PA-*ring if A is a* PA-*ring.*

THEOREM 2. *Let $\{f_{\alpha\mu}\colon A_\alpha \to A_\mu\}_{\alpha \le \mu}$ be a directed system of rings. If each A_α is a* BCS-*ring* (*resp. a* PAF-*ring or a* PA-*ring*) *then so is $\varinjlim A_\alpha$.*

PROOF OF THEOREM 1. The first assertion is proved as follows. Let $G = (c_{ij})\colon B^m \to B^n$ be a matrix having unit content, say $1 = \sum_{i,j} r_{ij} c_{ij}$ for some $r_{ij} \in B$. By (2), there exist $s_{ij}, d_{ij} \in A$ such that $f(s_{ij})$ is arbitrarily close to r_{ij} and $f(d_{ij})$ is arbitrarily close to c_{ij}. Thus by (1), we can choose s_{ij}, d_{ij} such that $f(\sum_{i,j} s_{ij} d_{ij}) = \sum_{i,j} f(s_{ij}) f(d_{ij})$ is arbitrarily close to $\sum r_{ij} c_{ij} = 1$. Thus we can bring $f(\sum_{i,j} s_{ij} d_{ij})$ sufficiently close to 1 to ensure (by (3)) that $\sum_{i,j} s_{ij} d_{ij}$ is a unit. Therefore $G' = (d_{ij})$ has unit content. Since A is a BCS ring, there is a rank one projective summand P in the image of G': that is, denoting by Π the canonical projection, the following sequence is exact

$$A^m \xrightarrow{\Pi \cdot G'} P \to 0.$$

Then the sequence

$$B^m \xrightarrow{(\Pi\otimes 1_B)\cdot f(G')} P\otimes B \to 0$$

is exact and the map $(\Pi\otimes 1_B)\cdot G$ is sufficiently close to $(\Pi\otimes 1_B)\cdot f(G')$. Thus, by [**5**, Lemma 1.3], the map $\Pi\otimes 1_B\cdot G$ is surjective and this gives a rank one projective summand in the image of G. It follows that B is a BCS-ring.

The proof of the second assertion is an adaptation of the above proof: one uses (SBI) to show that each reachable B-system can be approximated arbitrarily closely by a reachable A-system and then one reasons as above.

PROOF OF THEOREM 2. A proof of the BCS portion of the theorem goes as follows. Let $G\in M_{n\times m}(A)$ have unit content. Write $G=(a_{ij})$, then there exist $b_{ij}\in A$ so that $\sum_{i,j}a_{ij}b_{ij}=1$. If $f_\alpha\colon F_\alpha\to \varinjlim F_\alpha$ are the universal mappings, we can find $\lambda\in I$ such that there exist $c_{ij},d_{ij}\in A_\lambda$ so that $f_\lambda(c_{ij})=a_{ij}$ and $f_\lambda(d_{ij})=b_{ij}$. Then $f_\lambda((\sum_{i,j}c_{ij}d_{ij})-1)=0$, so there exists a $\mu\geq\lambda$ so that $f_{\lambda\mu}((\sum_{i,j}c_{ij}d_{ij})-1)=0$. Then $\sum_{i,j}f_{\lambda\mu}(c_{ij})f_{\lambda\mu}(d_{ij})=1$. Let $G_\mu=(f_{\lambda\mu}(c_{ij}))\in M_{n\times m}(A_\mu)$. Then G_μ has unit content, and $f_\mu(G_\mu)=G$. Since A_μ is BCS, there exists a matrix $B_\mu\in M_{m\times k}(A_\mu)$ such that $G_\mu B_\mu$ is a $*$-matrix. Let $B=f_\mu(B_\mu)$. Then $f_\mu(G_\mu B_\mu)=GB$ is a $*$-matrix over A.

The PAF and CAF proofs are completely analogous, but the PA assertion requires some formalism involving direct limits of categories. This is also easily adapted from [**5**, pp. 214–215].

These results have the following

COROLLARY. *Let X be a compact Hausdorff space of dimension ≤ 1. Then $C(X)$ is a BCS-ring.*

PROOF. By [**3**, p. 278], $X=\lim X_\alpha$, where $\{X_\alpha\}$ is an inverse system of polyhedra of dimension ≤ 1. Then by [**5**, p. 215], the ring morphism $f\colon \varinjlim C(X_\alpha)\to C(X)$ satisfies (1)–(3) and so by Theorems 1 and 2 we would be finished if we knew that $C(X_\alpha)$ were a BCS-ring for each α. But by [**5**, Theorem 6.3], for each α there exist a noetherian subring B_α of $C(X_\alpha)$ having dimension ≤ 1. Moreover, the inclusion $B_\alpha\subset C(X_\alpha)$ satisfies (1)–(3). By [**6** or **7**], B_α is a BCS-ring and another application of Theorem 1 completes the proof.

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