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COMPACT PARALLELIZABLE FOUR DIMENSIONAL SYMPLECTIC AND COMPLEX MANIFOLDS

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ABSTRACT. Examples of compact symplectic manifolds with no complex and/or Kähler structures are presented.

1. Introduction. Many examples of compact symplectic manifolds that carry no positive definite Kähler metric are now known. Here we present some compact 4-dimensional manifolds that have symplectic structures but carry no *complex* structures. More generally we prove

THEOREM 1.1. *Let $\mathbf{E}^4$ be a principal circle bundle over $\mathbf{E}^3$, which in turn is a principal circle bundle over a torus $\mathbf{T}^2$, so that the first Betti number of $\mathbf{E}^4$ satisfies $2 \leq b_1(\mathbf{E}^4) \leq 4$. Then*

(i) *if $b_1(\mathbf{E}^4) = 2$ then $\mathbf{E}^4$ has symplectic but no complex structures*;

(ii) *if $b_1(\mathbf{E}^4) = 3$ then $\mathbf{E}^4$ has both symplectic and complex structures but no positive definite Kähler metrics; however $\mathbf{E}^4$ carries indefinite Kähler metrics*;

(iii) $b_1(\mathbf{E}^4) = 4$ *if and only if $\mathbf{E}^4$ is a 4-torus $\mathbf{T}^4$.*

REMARKS. (1) Apparently the manifolds that occur in part (i) of Theorem 1.1 are the first examples of compact *symplectic* manifolds with no complex structures. Van de Ven [**VdV**], Yau [**Ya**] and Brotherton [**Br**] have given examples of compact 4-dimensional almost complex manifolds with no complex structures. Brotherton used Massey products to prove the nonexistence of complex structures on certain parallelizable 4-dimensional manifolds.

(2) Thurston [**Th**] has given an example of a compact symplectic manifold with no positive definite Kähler metric. (See also [**Ab, CFG, CFL, We1**].) In §3 we shall see that it is covered under part (ii) of Theorem 1.1. It is interesting to note that this example already occurs in the work of Kodaira [**Kod**, Theorem 19]. An explicit description of the Kodaira-Thurston example as a complex manifold is given in §3.

(3) The spaces $\mathbf{E}^4$ are all real parallelizable (but only $\mathbf{T}^4$ is complex parallelizable in the sense of Wang [**Wa**]). By a blowing up procedure one can construct *nonparallelizable* symplectic manifolds with no complex structure and/or positive definite Kähler metric [**Go**].

(4) Most of the manifolds considered in Theorem 1.1 have explicit matrix realizations as nilmanifolds [**CM**]. See also [**PS**], where it is proved that a compact manifold is a principal torus bundle over a torus if and only if it is a 2-step nilmanifold. The paper [**BG**] is also relevant.

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(5) As a corollary to part (ii) of Theorem 1.1, we observe that none of the complex structures mentioned there can be calibrated (in the sense of [**Gro**]) by a symplectic form, since otherwise the corresponding $\mathbf{E}^4$ would admit a positive definite Kähler metric.

(6) For a general discussion of symplectic manifolds constructed as fiber bundles see [**We2**].

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2. The topology of a principal circle bundle over a principal circle bundle over a torus. The classification of principal circle bundles is well known:

THEOREM 2.1 [**Kob**, p. 35, **Kos**, p. 133]. *There is a one-to-one correspondence between equivalence classes of principal circle bundles over a manifold* $\mathbf{M}$ *and the cohomology group* $H^2(\mathbf{M}, \mathbf{Z})$. *Furthermore, given an integral closed* 2*-form* Φ *on* $\mathbf{M}$, *there is a principal circle bundle* $\pi\colon \mathbf{E} \to \mathbf{M}$ *with connection form* η *such that* Φ *is the curvature of* η *(that is,* $\pi^*(\Phi) = d\eta$*).*

Now let α and β be integral closed 1-forms on $\mathbf{T}^2$ such that α and β are everywhere linearly independent and the cohomology class $[\alpha \wedge \beta]$ generates $H^2(\mathbf{T}^2, \mathbf{Z})$. Theorem 2.1 implies that for every integer n there is a principal circle bundle $\mathbf{E}^3 \to \mathbf{T}^2$ corresponding to $n[\alpha \wedge \beta]$ and a connection form γ on $\mathbf{E}^3$ chosen so that the curvature of γ is $n\alpha \wedge \beta$. The real minimal model of $\mathbf{E}^3$ is thus

$$M(\mathbf{E}^3) = \{\alpha, \beta, \gamma \mid d\alpha = d\beta = 0, d\gamma = n\alpha \wedge \beta\}.$$

(We use the same notation for differential forms on base spaces and their pullbacks to total spaces.) Then $H^1(\mathbf{E}^3, \mathbf{R}) = \{[\alpha], [\beta]\}$ and $H^2(\mathbf{E}^3, \mathbf{R}) = \{[\alpha \wedge \gamma], [\beta \wedge \gamma]\}$ when $n \neq 0$.

If $n = 0$, $\mathbf{E}^3$ is a 3-torus; otherwise $\mathbf{E}^3$ can be realized as the compact quotient $\Gamma_n \backslash \mathbf{H}_n$ where $\mathbf{H}_n$ is the Lie group of matrices of the form

$$\begin{pmatrix} 1 & a & -c/n \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

and Γ_n is the subgroup of $\mathbf{H}_n$ consisting of those matrices for which a, b and c are integers.

Principal circle bundles $\mathbf{E}^4 \to \mathbf{E}^3$ are classified by $H^2(\mathbf{E}^3, \mathbf{Z})$. When $n \neq 0$ the Gysin sequence yields $H^2(\mathbf{E}^3, \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_{|n|}$; thus for each pair of integers (p, q) there is a principal circle bundle corresponding to the class $p[\alpha \wedge \gamma] + q[\beta \wedge \gamma]$. Again we use Kobayashi's Theorem 2.1 to conclude that the connection form η of $\mathbf{E}^4 \to \mathbf{E}^3$ can be chosen so that its curvature form is precisely $p\alpha \wedge \gamma + q\beta \wedge \gamma$. It follows that when $n \neq 0$ the (real) minimal models of the $\mathbf{E}^4$ are given as follows:

$$M(\mathbf{E}^4) = \{\alpha, \beta, \gamma, \eta \mid d\alpha = d\beta = 0,\ d\gamma = n\alpha \wedge \beta, d\eta = p\alpha \wedge \gamma + q\beta \wedge \gamma\}.$$

Clearly, $b_1(\mathbf{E}^4) = 3$ if $p = q = 0$ and $b_1(\mathbf{E}^4) = 2$ otherwise. The case $n = 0$ is similar: for each triple of integers (p, q, r) there is an $\mathbf{E}^4$ with

$$M(\mathbf{E}^4) = \{\alpha, \beta, \gamma, \eta \mid d\alpha = d\beta = d\gamma = 0, d\eta = r\alpha \wedge \beta + p\alpha \wedge \gamma + q\beta \wedge \gamma\}.$$

Then $b_1(\mathbf{E}^4) = 4$ if $r = p = q = 0$ and $b_1(\mathbf{E}^4) = 3$ otherwise.

LEMMA 2.2. *The minimal model $M(\mathbf{E}^4)$ is not formal if one of n, p, q, r is different from zero.*

PROOF. It suffices to find nonzero Massey products. Suppose $n \neq 0$ (the proof when $n = 0$ is simpler). Then the cohomology classes $[\alpha \wedge \alpha]$ and $[\alpha \wedge n\beta]$ are both zero, so that the Massey product $\langle [\alpha], [\alpha], [n\beta] \rangle$ is well defined. By definition it is represented by $\alpha \wedge \gamma$. Now $[\alpha \wedge \gamma] \neq 0$ for $\mathbf{E}^3$; it is also nonzero in the cohomology of $\mathbf{E}^4$ except when $p \neq 0$, $q = 0$. But in this case the Massey product $\langle [\beta], [\beta], [n\alpha] \rangle$ is nonzero.

3. Proof of Theorem 1.1. First let us note that in all cases $\mathbf{E}^4$ has many symplectic forms. For example

$$\Omega = (a\alpha + b\beta) \wedge \gamma + (e\alpha + f\beta) \wedge \eta$$

is closed if a, b, e, f are constants such that $fp - eq = 0$, and has maximal rank if $af - be \neq 0$.

PROOF OF (i). We use [**Kod**, Theorem 25]: *A [complex] surface is a deformation of an algebraic surface if and only if its first Betti number is even.* Suppose $\mathbf{E}^4$ with $b_1(\mathbf{E}^4) = 2$ had a complex structure. Then [**Kod**, Theorem 25] would imply that $\mathbf{E}^4$ would have a positive definite Kähler metric. But now a result of [**DGMS**] would imply that $M(\mathbf{E}^4)$ is formal, and this is impossible by Lemma 2.2.

REMARK. It is amusing to compare an $\mathbf{E}^4$ with $b_1(\mathbf{E}^4) = 2$ with the Kähler manifold $\mathbf{S}^2 \times \mathbf{T}^2$. Both are parallelizable and have the same Betti numbers. But $\mathbf{E}^4$ has nonzero Massey products while $\mathbf{S}^2 \times \mathbf{T}^2$ does not.

PROOF OF (ii). When $b_1(\mathbf{E}^4) = 3$ and $n \neq 0$ an explicit complex structure on $\mathbf{E}^4$ can be constructed as follows. Let X, Y, Z, T be the parallelization dual to $\alpha, \beta, \gamma, \eta$; the only nonzero bracket is $[X, Y] = -nZ$. Now define an almost complex structure J on $\mathbf{E}^4$ by $JX = Y$, $JZ = T$. A direct calculation shows that the Nijenhuis tensor of J vanishes; consequently J is complex. A similar construction yields a complex structure on an $\mathbf{E}^4$ with $n = 0$.

None of these $\mathbf{E}^4$ can possess a positive definite Kähler metric since $b_1(\mathbf{E}^4)$ is odd. (There are also nonzero Massey products.) Nonetheless an indefinite Kähler metric ϕ for the complex structure J can be constructed as follows. Let Ω be a symplectic form which has type $(1,1)$ with respect to J; for example we can take $\Omega = \alpha \wedge \gamma + \beta \wedge \eta$. Then put $\phi(U, V) = \Omega(U, JV)$ for vector fields U, V on $\mathbf{E}^4$.

In general suppose that Ω is Hermitian with respect to an almost complex structure J so that the metric ϕ is given by $\phi(x, y) = \Omega(x, Jy)$. For vector fields X, Y, Z we have that

$$2\nabla_X(\Omega)(Y, Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) - \phi(X, S(Y, JZ)),$$

where S denotes the Nijenhuis tensor of J [**Gra**, formula (4.8)]. It follows that if J is integrable and Ω is symplectic, then ϕ is Kählerian, but possibly indefinite.

REMARK. The Kodaira-Thurston example belongs to case (ii); explicitly it is $\Gamma_{-1} \backslash H_{-1} \times \mathbf{S}^1$. As a complex manifold it has the following description. For each Gaussian integer n let

$$\mathbf{G}_n = \left\{ \left. \begin{pmatrix} 1 & \bar{z} & w/n \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right| z \text{ and } w \text{ are complex} \right\}.$$

$\mathbf{G}_n$ is a complex manifold and as a Lie group it is left holomorphic but not right holomorphic. Let Ψ_n be the subgroup of $\mathbf{G}_n$ consisting of all those matrices whose elements are Gaussian integers. Then $\mathbf{E}^4 = \Psi_n \backslash \mathbf{G}_n$ is a nilmanifold and a complex manifold (but not a complex nilmanifold). The Kodaira-Thurston example is $\Psi_1 \backslash \mathbf{G}_1$.

The proof of (iii) is obvious.

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