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AN EXTENSION OF THE CLOSED UNBOUNDED FILTER

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ABSTRACT. A natural extension of the closed unbounded filter is introduced. This extension coincides with the closed unbounded filter on uncountable, regular cardinals κ, but in general does not for $P_\kappa\lambda$ and $[\lambda]^\kappa$.

Henceforth, κ will be a regular, uncountable cardinal, unless specified otherwise. A closed unbounded subset of κ is a cofinal subset of κ which contains the supremum of all increasing sequences from the subset of length less than κ. The collection of all closed unbounded subsets of κ generates a κ-complete, normal filter over κ called the club filter. The notion of a closed unbounded subset of a cardinal κ has been generalized to the set of all subsets of λ of cardinality less than κ, $P_\kappa\lambda$ (see [**3**]), and the set of all subsets of λ of cardinality κ, $[\lambda]^\kappa$ (see [**2**]). In both instances the collection of closed unbounded sets generate κ-complete, fine, normal filters, the club filters. This paper introduces a natural extension of the club filter. This extension coincides with the club filter on κ, but in general does not for $P_\kappa\lambda$ and $[\lambda]^\kappa$.

The motivation for the filter arose from the desirable property of certain sequences (or proper chains in the case of $P_\kappa\lambda$) $\{p_\alpha\colon \alpha<\delta\}$ which satisfy

$$\left|\bigcup_{\alpha<\delta} p_\alpha\right| = \bigcup_{\alpha<\delta} |p_\alpha|,$$

where $|A|$ denotes the cardinality of A. Since

$$\left|\bigcup_{\alpha<\delta} p_\alpha\right| = |\delta| \bigcup_{\alpha<\delta} |p_\alpha|$$

this property is satisfied by sequences (or chains) $\{p_\alpha\colon \alpha<\delta\}$ which satisfy

$$|\delta| \leq \bigcup_{\alpha<\delta} |p_\alpha|.$$

A canonical example of a sequence which fails to satisfy this is $\{p_\alpha\colon \alpha<\delta\}$ where $p_\alpha=\alpha$ and $\delta=\omega_1$.

DEFINITION. A subset "b" of κ is said to be L-closed if for any sequence $\{p_\alpha\colon \alpha<\delta\}\subset b$ with $\delta<\kappa$ and $|\delta|\leq\bigcup_{\alpha<\delta}|p_\alpha|$, then $\bigcup_{\alpha<\delta}p_\alpha\in b$.

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An L-closed, unbounded subset of κ is a cofinal subset of κ which is L-closed. The verification that the L-closed, unbounded subsets of κ generate a κ-complete, normal filter over κ, called the L-club filter, is a routine exercise once familiar with the details of generating the club filter over κ (see [**4**]). The L-club filter and the club filter coincide on κ: Let $b \subset \kappa$ be closed unbounded. Then b is L-closed unbounded. And if $b \subset \kappa$ is L-closed unbounded, consider

(i) if κ is a limit cardinal

$$a = b \cap \{\alpha < \kappa \colon \alpha \text{ is a cardinal}\}.$$

(ii) if κ is a successor cardinal and $\kappa = \gamma^+$

$$a = (b \cap \kappa - \gamma).$$

In both cases $a \subset b$ and a is closed unbounded.

For $X \subset P_\kappa\lambda$, X is unbounded if for any $x \in P_\kappa\lambda$, then there exists a $y \in X$ such that $x \subset y$. And X is closed if for any collection $\{p_\alpha \colon \alpha < \delta\} \subset X$ with $p_\alpha \subset p_{\alpha+1}$ for $\alpha < \delta$ (called a chain of subsets from $P_\kappa\lambda$) with $\delta < \kappa$, then $\bigcup_{\alpha<\delta} p_\alpha \in X$.

DEFINITION. A set $X \subset P_\kappa\lambda$ is said to be L-closed if whenever $\{p_\alpha \colon \alpha < \delta\} \subset X$ is a proper chain such that $|\delta| \leq \bigcup_{\alpha<\delta} |p_\alpha| < \kappa$, then $\bigcup_{\alpha<\delta} p_\alpha \in X$.

Let

$$L_\kappa\lambda = \{A \subset P_\kappa\lambda \colon \text{ there exists an } X \subset A \text{ which is } L\text{-closed unbounded}\}.$$

PROPOSITION 1. *$L_\kappa\lambda$ is a κ-complete, fine, normal filter over $P_\kappa\lambda$.*

PROOF. The proof follows closely the proof that the closed unbounded sets on $P_\kappa\lambda$ generate the club filter (see [**3**]). The only modification for this proof requires that wherever a chain $\{p_\alpha \colon \alpha < \delta\}$ is constructed in the club filter proof, the chain must be made to satisfy $|\delta| \leq \bigcup_{\alpha<\delta} |p_\alpha|$. This presents little difficulty in any of the situations where chains are needed. The proof of the next proposition will show the ease at which a chain, in most of the circumstances required here, can be made to satisfy this condition.

A subset $D \subset P_\kappa\lambda$ is said to be directed if given $x, y \in D$, then there exists $z \in D$ such that $x \cup y \subset z$. A subset $X \subset P_\kappa\lambda$ is said to be closed under directed sets if given $D \subset X$ such that $|D| < \kappa$ and D is directed, then $\bigcup D \in X$. It is a result of Solovay that X is closed unbounded if and only if X is closed under directed sets (see [**5**]). The analog here is the following:

PROPOSITION 2. *An L-closed unbounded subset of $P_\kappa\lambda$ is closed under unions of directed sets D where $|D| \leq \bigcup\{|p| \colon p \in D\}$.*

PROOF. (This is basically Solovay's proof with the required modifications.) Let B be an L-closed unbounded subset of $P_\kappa\lambda$ and $D \subset B$ a directed set such that $|D| \leq \bigcup\{|p| \colon p \in D\}$. Only the case where $|D| > \aleph_0$ requires an adjustment. Assume if D' is any directed subset of B, $|D'| < |D|$ and $|D'| \leq \bigcup\{|p| \colon p \in D\}$, then $\bigcup D' \in B$.

CLAIM. Given any set $X \subset D$, there exists a set X^+ such that

(1) $X \subset X^+ \subset D$,

(2) $|X^+| \leq |X| + \aleph_0 \leq \bigcup\{|p| \colon p \in X^+\}$ and

(3) X^+ is directed.

PROOF OF CLAIM. See [**5**].

Well order D by $p_0, p_1, \ldots, p_\alpha, \ldots$ where $\alpha < |D|$. Let

$$D_0 = \{p_0\},$$
$$D_1 = \{D_0, p_1\}^+,$$
$$D_2 = \{D_0, D_1, p_2, q_2\}^+ \text{ where } q_2 \in D \text{ such that } |q_2| \geq 2,$$
$$\vdots$$
$$D_\alpha = \{\{D_\beta\}_{\beta<\alpha}, p_\alpha, q_\alpha\}^+,$$

where $q_\alpha \in D$ such that $|q_\alpha| \geq \alpha + \aleph_0$. (Note: such a q_α exists since $|D| \leq \bigcup\{|p| \colon p \in D\}$.) Now,

$$|D_\alpha| = |\{\{D_\beta\}_{\beta<\alpha}, p_\alpha, q_\alpha\}^+| \leq \alpha + \aleph_0 < |D|.$$

Hence,

$$|D_\alpha| \leq \bigcup\{|p| \colon p \in D_\alpha\}.$$

By our assumption, $\bigcup D_\alpha \in B$. So let

$$Q_\alpha = \bigcup D_\alpha \in B \quad \text{for all } \alpha < |D|.$$

Then $\{Q_\alpha \colon \alpha < |D|\} \subset B$ is a chain and

$$\begin{aligned}|D| &\leq \bigcup\{\alpha+ \colon \alpha < |D|\} \leq \bigcup\{|q_\alpha| \colon \alpha < |D|\}\\ &\leq \bigcup\{|UD_\alpha| \colon \alpha < |D|\} = \bigcup\{|Q_\alpha| \colon \alpha < |D|\}.\end{aligned}$$

Since B is L-closed, $\bigcup D = \bigcup_{\alpha<|D|} Q_\alpha \in B$.

In contrast to the situation on cardinals where the L-club and club filters coincide, the next proposition shows that in general this not the case for $P_\kappa\lambda$ when a large cardinal assumption is made on κ.

PROPOSITION 3. *Assume κ is a huge cardinal and λ is any cardinal greater than κ such that there exists a κ-complete, fine, normal ultrafilter over $[\lambda]^\kappa$, then there exists an L-closed unbounded subset of $P_\kappa\lambda$ which is not in the club filter.*

PROOF. Since λ must be measurable (see [**1** or **6**]), a regular cardinal γ can be chosen such that $\kappa < \gamma < \lambda$. Now λ must be a regular cardinal so λ can be partitioned into λ-many disjoint intervals of length γ. For $\alpha < \lambda$ denote the αth such interval as γ_α. Given any $\beta < \lambda$ let the γth index of β, denoted $\gamma(\beta)$, be

$$\Gamma(\beta) = \text{ order type of } \gamma_\alpha \cap \beta, \quad \text{where } \beta \in \gamma_\alpha.$$

For $x \in P_\kappa\lambda$ let

$$\Gamma'' x = \{\Gamma(\beta) \colon \beta \in x\}.$$

Set

$$B = \{x \in P_\kappa\lambda \colon |\Gamma'' x| = |x|\}.$$

CLAIM 1. B is L-closed unbounded.

PROOF OF CLAIM. B is readily seen to be unbounded, since for any $x \in P_\kappa\lambda$, $x \cup |x| \cup \omega \in B$.

Next, let $\{p_\alpha \colon \alpha < \delta\} \subset B$ be a chain of size $\delta < \kappa$, where $|\delta| \leq \bigcup_{\alpha<\delta} |p_\alpha|$. Suppose $|\bigcup_{\alpha<\delta} p_\alpha| > |\Gamma'' \bigcup_{\alpha<\delta} p_\alpha|$. Let $\beta = |\Gamma'' \bigcup_{\alpha<\delta} p_\alpha|$. So $|\bigcup_{\alpha<\delta} p_\alpha| > \beta$.

Since $\bigcup_{\alpha<\delta}|p_\alpha| = |\bigcup_{\alpha<\delta} p_\alpha| > \beta$ there exists $\alpha < \delta$ such that $|p_\alpha| > \beta$. But $|\Gamma'' p_\alpha| = |p_\alpha|$, which is clearly false. Therefore, $\bigcup_{\alpha<\gamma} p_\alpha \in B$, and the claim is proved.

DiPrisco and Marek used the following type of construction to define their notion of closed unbounded sets on $[\lambda]^\kappa$ (see [**2**]):

For $B \subset P_\kappa\lambda$ from above, set

$$A_B = \left\{ p \in [\lambda]^\kappa \colon \text{ there exists a chain } \{p_\alpha \colon \alpha < \kappa\} \subset B \text{ and } p = \bigcup_{\alpha<\kappa} p_\alpha \right\}.$$

CLAIM 2. If $p \in A_B$, then $|\Gamma'' p| = \kappa$.

PROOF OF CLAIM. Assume $|\Gamma'' p| < \kappa$. Since $\kappa = |p| = |\bigcup_{\alpha<\kappa} p_\alpha| = \bigcup_{\alpha<\kappa}|p_\alpha|$ there exists $\alpha < \kappa$ such that $|p_\alpha| > |\Gamma'' p|$. But this gives

$$|\Gamma'' p_\alpha| = |p_\alpha| > |\Gamma'' p| = \left| \Gamma'' \bigcup_{\alpha<\kappa} p_\alpha \right|,$$

which cannot be true.

Finally, assume there exists a closed unbounded set C from $P_\kappa\lambda$ which is a subset of B. By a result in [**2**] if U is any κ-complete, fine, normal ultrafilter over $[\lambda]^\kappa$ and C is any closed unbounded subset of $P_\kappa\lambda$, then $A_C \in U$. So by our assumption, $A_B \in U$. That is, $\{p \in [\lambda]^\kappa \colon |\Gamma'' p| = \kappa\} \in U$.

Let $j \colon V \to M$ be the canonical elementary embedding produced by the ultrafilter U on $[\lambda]^\kappa$. Then, $A \in U$ iff $j''\lambda \in j(A)$. Hence,

$$M \models (j\Gamma)'' j''\lambda = j(\kappa)$$

by the above. However, the definition of Γ and the elementarity of j yield

$$M \models (j\Gamma)'' j''\lambda = j''\gamma.$$

But, this is a contradiction, since $\gamma < \lambda$.

This demonstrates that there are subsets of $P_\kappa\lambda$ which are L-closed unbounded but not closed. However, such subsets of $P_\kappa\kappa^+$ do not exist.

Whether or not an L-closed unbounded subset of $P_\kappa\lambda$ can be constructed which does not contain a closed unbounded subset, without first assuming the existence of a huge cardinal, is open. However, there exist examples of L-closed unbounded subsets of $P_\kappa\lambda$ which are not closed unbounded. The following was provided by C. A. DiPrisco:

$$E = \{p \in P_\kappa\lambda \colon |p \cap \kappa| = |p - \kappa|\}.$$

The problem remains to determine whether or not such sets are in the club filter on $P_\kappa\lambda$.

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