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# ISOMORPHISMS OF PRIME GOLDIE SEMIPRINCIPAL LEFT IDEAL RINGS

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ABSTRACT. A prime (left) Goldie semiprincipal left ideal ring is the endomorphism ring $E(F, A)$ of a free module $A$, of finite rank, over a (left) Ore domain $F$. We examine the uniqueness of the module $(F, A)$ in the sense of determining necessary and sufficient conditions that every isomorphism of $E(F, A)$ is induced by a semilinear module isomorphism of $(F, A)$.

**Introduction.** Let $K$ be a prime (left) Goldie ring in which each finitely generated left ideal is principal. Then $K$ is isomorphic to the endomorphsm ring $E(F, A)$ of a free module $A$, of finite rank, over a (left) Ore domain $F$ [**3**, **6**, **8**]. We examine the uniqueness of this representation in the following sense: What are necessary and sufficient conditions such that, whenever $K \cong E(F, A) \cong E(G, B)$ (with $A, B$ free and finitely generated over domains $F$ and $G$), every isomorphism of $E(F, A)$ upon $E(G, B)$ is induced by a (semilinear) module isomorphism of $(F, A)$ upon $(G, B)$? We are able to show (Theorem 3.1) that this is precisely when all minimal right annihilators of $K$ are mutually isomorphic (as right $K$ modules), or equivalently when $K$ is isomorphic to $E(F, A)$ with $F$, a semiprincipal left ideal domain. A key point is showing that $K$ is a Baer ring (Lemma 2.3).

If "semilinear module isomorphism" is understood to include mappings between extensions of the underlying modules $(F, A)$ and $(G, B)$ then every isomorphism of $E(F, A)$ onto $E(G, B)$ is induced by a semilinear module isomorphism (Theorem 3.2).

**1. Definitions and preliminaries.** All rings have identity elements. The left module $A$ over the ring $F$ will be denoted $(F, A)$ and its endomorphism ring (operating on the right of $A$) will be denoted $E(F, A)$. $(A, E)$ will denote the right $E$ module $A$. The statement that $(F, A, E)$ is a bimodule indicates that $A$ is to be considered as a left $F$ module and a right $E$ module.

The arguments of Baer [**1**, Proposition 5, p. 176] with minor modifications can be used to prove

LEMMA 1.1. *If there is an idempotent $e$ in $E = E(F, A)$ such that $Ae$ is free and cyclic, then the bimodule $(F, A, E)$ is isomorphic to the bimodule $(eEe, eE, E)$ in the following sense*:

(1) *$F$ and $eEe$ are isomorphic rings; $A$ and $eE$ are isomorphic additive groups, and there is a semilinear module isomorphism of the left modules $(F, A)$ and $(eEe, eE)$.*

(2) *There is an $E$ isomorphism of the right modules $(A, E)$ and $(eE, E)$.*

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LEMMA 1.2. *Let $(F, A)$ possess a free cyclic summand, where $F$ is a domain, and let $E = E(F, A)$ be its endomorphism ring. If $eE$ and $fE$ are $E$-isomorphic for each pair of primitive idempotents $e, f$ in $E$, then every indecomposable summand of $A$ is free and cyclic.*

PROOF. Let $S$ be an indecomposable summand of $A$, and $f$ an idempotent in $E$ such that $S = Af$. Let $e$ be an idempotent for which $Ae$ is free and cyclic. Then $e$ and $f$ are primitive since $Ae$ and $Af$ are indecomposable summands of $A$. By Lemma 1.1, there exists a bimodule isomorphism $\phi$ of $(F, A, E)$ onto $(eEe, eE, E)$. Hence $(A\sigma)\phi = (A\phi)\sigma = (eE)\sigma$, for each $\sigma \in E(F, A)$. If we let $\sigma = f$ then $S\phi = (Af)\phi = (eE)f = eEf$. Since $eE$ and $fE$ are $E$-isomorphic, it follows that $eEf$ is free and cyclic over $eEe$ [**7**, Lemma 2.3, p. 326]. Since $\phi$ is a (semilinear) module isomorphism of $(F, A)$ and $(eEe, eE)$, $S$ must be free and cyclic over $F$.

**2. Prime, Goldie, semi-pli rings.** A ring $K$ is a pli ring (semi-pli ring), if each left ideal (finitely generated left ideal) is a principal left ideal.

A nonzero left module is uniform if any two nonzero submodules have nonzero intersection.

We shall say $(F, A) \in \mathscr{F}$, if $F$ is a domain, and $(F, A)$ is a finitely generated free module. If $(F, A) \in \mathscr{F}$, then $E(F, A)$ is prime. Results of Goldie [**3**], Robson [**8**] and Jategaonkar [**6**] (see also [**4**, Chapter 4]) imply that

(1) If $(F, A) \in \mathscr{F}$, then $E(F, A)$ is a (left) Goldie ring if, and only if, $F$ is (left) Ore.

(2) If $K$ is a prime Goldie semi-pli ring, there exists a module $(F, A) \in \mathscr{F}$ such that $K \cong E(F, A)$.

The proof of Lemma 4.11 of Chapter 4 of [**4**] shows that $A$ may be chosen as any minimal right annihilator of $K$. (We do not need the specific identification of $F$.) If $\phi$ is the isomorphism of $K$ onto $E(F, A)$ then for $k \in K$, $xk^{\phi} = x \cdot k$, for each $x \in A$.

We need the following slight generalization of a result of Jategaonkar [**6**, Proposition 2.11, p. 51]. His arguments can be followed with minor modifications.

LEMMA 2.1. *Let $(F, A) \in \mathscr{F}$, with $A$ free on $n$ generators. If $E(F, A)$ is a Goldie semi-pli ring, then for every set $J_1, J_2, \ldots, J_n$ of nonzero finitely generated left ideals of $F$, we have $J_1 \oplus J_2 \oplus \cdots \oplus J_n$ is isomorphic (as a left $F$-module) to $(F, A)$.*

REMARK 1. If $E(F, A)$ satisfies the hypothesis of the lemma, each finitely generated left ideal of $F$ is projective. Hence each finitely generated submodule of $A$ is projective.

A ring in which each left annihilator ideal is a principal left ideal generated by an idempotent is called a Baer ring (the terminology is due to Kaplansky). The proof of the following is straightforward and will be omitted.

LEMMA 2.2. *Let $K$ be a Baer ring, and $e$ an idempotent in $K$. Then the following are equivalent:*

(1) *$e$ is primitive.*

(2) *$eK$ is a minimal right annihilator.*

(3) *$Ke$ is a minimal left annihilator.*

If $S$ is a submodule of $A$, $L(S)$ will denote the left ideal of $E(F, A)$ consisting of all $\sigma$ in $E(F, A)$ for which $A\sigma \subseteq S$. For $\tau$ in $E(F, A)$, $\mathscr{L}(\tau)$ will consist of all $\rho$ in $E(F, A)$ for which $\rho\tau = 0$, while $N(\tau)$ will be the submodule of $A$ consisting of all $a$ in $A$ for which $a\tau = 0$.

LEMMA 2.3. *If $K$ is a prime, Goldie, semi-pli ring, then $K$ is a Baer ring.*

PROOF. $K \cong E(F, A)$ where $A$ is free of finite rank $n$, over an Ore domain $F$. If $J$ is a left annihilator ideal of $E = E(F, A)$, then $J = \mathscr{L}(\sigma)$, $\sigma \in E$ [**2**, Theorem 3.7, p. 208]. But $\mathscr{L}(\sigma) = L[N(\sigma)]$ follows from the relevant definitions. We shall show that $N(\sigma)$ is a summand of $A$, from which it will follow that $L[N(\sigma)]$ is generated by an idempotent. Since $A$ is finitely generated, so is $A\sigma$. By Remark 1, following Lemma 2.1, $A\sigma$ is projective. Since $A\sigma \cong A/N(\sigma)$, it follows that $N(\sigma)$ is a summand of $A$. The fact that $L[N(\sigma)]$ is generated by an idempotent now follows as in the first part of the proof in [**1**, Proposition 1, p. 178].

## 3. The main theorems.

THEOREM 3.1. *Let $K$ be a prime left Goldie semi-pli ring. The following statements are equivalent:*

(1) *If $K \cong E(F, A)$, $K \cong E(G, B)$ with $(F, A), (G, B) \in \mathscr{F}$, then every isomorphism of $E(F, A)$ upon $E(G, B)$ is induced by a semilinear module isomorphism of $(F, A)$ upon $(G, B)$.*

(2) *All minimal right annihilators of $K$ are isomorphic (as right $K$ modules).*

(3) *All minimal left annihilators of $K$ are isomorphic (as left $K$ modules).*

(4) *All finitely generated uniform left ideals of $K$ are isomorphic (as left $K$ modules).*

(5) *There exists a semi-pli domain $F$, and a module $(F, A) \in \mathscr{F}$ such that $K \cong E(F, A)$.*

PROOF. (1)$\Rightarrow$(2) Let $J_1, J_2$ be minimal right annihilators of $K$, so that $J_1 = eK$, $J_2 = fK$ with $e, f$ primitive idempotents of $K$ (Lemmas 2.2, 2.3). By item (2) preceding Lemma 2.1, there is an isomorphism $\alpha$ of $K$ onto $E(F, eK)$ such that if $k \in K$, $xk^\alpha = x \cdot k$, for all $x \in eK$, and an isomorphism $\beta$ of $K$ onto $E(G, fK)$ such that if $k \in K$, $yk^\beta = y \cdot k$ for all $y \in fK$.

Clearly $\phi = \alpha^{-1}\beta$ is an isomorphism of $E(F, eK)$ onto $E(G, fK)$. For each $k \in K$, $k^\alpha = \sigma \in E(F, eK)$, so that

$$\sigma^\phi = (k^\alpha)^{\alpha^{-1}\beta} = k^\beta \in E(G, fK).$$

By assumption $\phi$ is induced by a semilinear isomorphism $\rho$ of $(F, eK)$ upon $(G, fK)$. In particular,

$$y\sigma^\phi = y(\rho^{-1}\sigma\rho), \quad \text{for each } y \in fK,$$

or letting $y = x\rho$, for $x \in eK$, we have

$$(x\rho)\sigma^\phi = x(\sigma\rho) = (x\sigma)\rho, \quad \text{for each } x \in eK, \quad \text{or} \quad (x\rho)k^\beta = (xk^\alpha)\rho.$$

Hence

$$(x\rho) \cdot k = (x \cdot k)\rho, \quad \text{for each } x \in eK,$$

so that $\rho$ is a $K$-isomorphism of the right annihilators $eK$ and $fK$.

(2) $\Rightarrow$ (5) By items (1) and (2) preceding Lemma 2.1, $K \cong E(F, A)$ with $(F, A) \in \mathscr{F}$ and $F$, a left Ore domain. Now let $J$ be a nonzero finitely generated left ideal of $F$. Since $F$ is an Ore domain, $J$ is an indecomposable $F$ module. By Lemma 2.1, there exists an $F$-module $Q$ such that $(F, J \oplus Q) \cong (F, A)$. Under this isomorphism $J$ maps onto an indecomposable summand of $A$, which is free and cyclic by Lemma 1.2. Hence $J$ must be a principal left ideal.

To prove (5) $\Rightarrow$ (1), Let $(F, A), (G, B) \in \mathscr{F}$ with $K \cong E(F, A) \cong E(G, B)$. Let $K \cong E(F_0, A_0)$ with $(F_0, A_0) \in \mathscr{F}$, and $F_0$ a semi-pli domain. Then of course $E(F_0, A_0) \cong E(F, A)$. By results of Jategaonkar [**6**, Theorem 1.6, p. 37, Proposition 1.8, p. 40, and Lemma 2.8, p. 49] this latter isomorphism is induced by a (semilinear) module isomorphism of $(F_0, A_0)$ onto $(F, A)$. In particular, $F$ is a semi-pli domain. Now we can apply the same argument to the isomorphism of $E(F, A)$ and $E(G, B)$ to conclude that it is induced by a semilinear module isomorphism of $(F, A)$ upon $(G, B)$.

(2) $\Leftrightarrow$ (3) If $e, f$ are idempotents in any ring $K$, then $eK \cong fK$ if, and only if, $Ke \cong Kf$ [**5**, Corollary, p. 51].

(4) $\Leftrightarrow$ (5) is a result of Robson [**8**, Theorem 5.3, p. 627].

REMARK 1. The restriction that $F$ and $G$ be domains is essential. Let $F$ be a pli domain, and $G = E(F, F^{(n)})$, $n > 1$. Let $B = G^{(m)}$, $m > 1$, and $A = F^{(mn)}$. Then $E(F, A)$ and $E(G, B)$ are isomorphic prime, pli rings, with $F, G$, nonisomorphic pli rings.

REMARK 2. Swan [**9**, Lemma 1, p. 57] gives an example of an $(F', A') \in \mathscr{F}$ for which $E(F', A')$ is a pli ring, but $F'$ is not a pli domain.

Theorem 3.1 then implies that if $K = E(F', A')$ is the ring of Swan's example there exist $(F, A)$ and $(G, B) \in \mathscr{F}$, with $K \cong E(F, A), K \cong E(G, B)$ but with an isomorphism of $E(F, A)$ upon $E(G, B)$ which is not induced by a semilinear module isomorphism of $(F, A)$ upon $(G, B)$. In fact it is shown in [**10**] that there exist $(F, A), (G, B) \in \mathscr{F}$ with $K \cong E(F, A) \cong E(G, B)$ but $F \not\cong G$, so that there is no (semilinear) module isomorphism of $(F, A)$ upon $(G, B)$. Nevertheless, even in this case, every isomorphism of $E(F, A)$ upon $E(G, B)$ is induced by a (semilinear) module isomorphism if we are willing to include mappings between extensions of the modules $(F, A)$ and $(G, B)$.

THEOREM 3.2. *Let $K$ be a prime (left) Goldie semi-pli ring, such that $K \cong E(F, A)$, $K \cong E(G, B)$ with $(F, A), (G, B) \in \mathscr{F}$. Then, there exist modules $(\overline{F}, \overline{A})$, $(\overline{G}, \overline{B})$, which are extensions of $(F, A)$ and $(G, B)$ respectively, such that every isomorphism of $E(F, A)$ upon $E(G, B)$ is induced by a semilinear module isomorphism of $(\overline{F}, \overline{A})$ upon $(\overline{G}, \overline{B})$.*

PROOF. Assume that $(F, A), (G, B) \in \mathscr{F}$, and $E(F, A)$ and $E(G, B)$ are isomorphic Goldie semi-pli rings. Since $F$ and $G$ are Ore domains, they have quotient division rings $\overline{F}$ and $\overline{G}$. Let $\overline{A} = \overline{F} \otimes_F A$, and $\overline{B} = \overline{G} \otimes_G B$. Then $\overline{A}$ and $\overline{B}$ are (left) vector spaces over $\overline{F}$ and $\overline{G}$ respectively and with the usual identification, $(F, A)$ and $(G, B)$ are embedded in $(\overline{F}, \overline{A})$ and $(\overline{G}, \overline{B})$, and $E(F, A)$ and $E(G, B)$ are embedded as (left) orders in $E(\overline{F}, \overline{A})$ and $E(\overline{G}, \overline{B})$ respectively [**6**, Proposition 2.14, p. 20]. If $\phi$ is an isomorphism of $E(F, A)$ upon $E(G, B)$, it has a unique extension to $\overline{\phi}$ an isomorphism of $E(\overline{F}, \overline{A})$ upon $E(\overline{G}, \overline{B})$. Hence there exists a one-one semilinear transformation $\rho$ of $(\overline{F}, \overline{A})$ onto $(\overline{G}, \overline{B})$ such that $\sigma^{\overline{\phi}} = \rho^{-1}\sigma\rho$

for each $\sigma \in E(\overline{F}, \overline{A})$ [**1**, Theorem 1, p. 183]. In particular $\sigma^{\phi} = \rho^{-1}\sigma\rho$ for each $\sigma \in E(F, A)$.

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