

ORDER RELATION IN QUADRATIC JORDAN RINGS AND A STRUCTURE THEOREM

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ABSTRACT. It is shown that the relation defined by $x \leq y$ if and only if $V_x x = V_x y$ and $U_x x = U_x y = U_y x$ is an order relation for quadratic Jordan algebras without nilpotent elements, which extends our previous one for linear Jordan algebras, and reduces to the usual Abian order for associative algebras. We prove that a quadratic Jordan algebra is isomorphic to a direct product of division algebras if and only if the algebra has no nilpotent elements and is hyperatomic and orthogonally complete.

Preliminaries. Throughout this paper, Φ will denote a commutative associative ring with identity element and $J = (J, U, 1)$ a quadratic unital Jordan algebra over Φ , that is, J is a Φ -module, 1 an element of J , and $U_x y$ is a product quadratic in x and linear in y which satisfies certain axioms analogous to properties of the product xyx in associative algebras. We use the notation $V_{x,y} z = U_{x,z} y = \{xyz\}$ for the polarization $U_{x,z} = U_{x+z} - U_x - U_z$, $V_x y = U_{x,1} y = U_{x,y} 1 = x \circ y$.

If A is a unital associative algebra, then A^+ becomes a quadratic Jordan algebra via

$$U_x y = xyx, \quad V_{x,y} z = xyz + zyx, \quad V_x y = xy + yx.$$

Jordan powers x^n can be defined recursively (corresponding to the usual associative powers in A^+); we will need only powers $x^2 = U_x 1$ and $x^3 = U_x x$. J has no nilpotent elements ($x^n = 0$ for some n forces $x = 0$) if and only if $x^2 = 0$ or $x^3 = 0$ forces $x = 0$. MacDonald's principle asserts that any Jordan relation $f(x, y, z) = 0$ in three variables which is of degree at most 1 in z , and which holds in all associative algebras A^+ , necessarily holds in all Jordan algebras. Any Jordan identity that we use will be a consequence of this principle.

1. Orthogonality.

DEFINITION. x_1 and x_2 are orthogonal, $x_1 \perp x_2$, if $x_1 \circ x_2 = U_{x_1} x_2 = U_{x_2} x_1 = 0$.

This relation is symmetric in x_1 and x_2 . Zero is orthogonal to everything; if J has no nilpotent elements, then $x \perp x$ forces $x = 0$. If $J \subset A^+$ is special, then associative orthogonality $x_1 x_2 = x_2 x_1 = 0$ is sufficient (and, if A has no nilpotent elements, necessary: $(x_i x_j)^2 = x_i (U_{x_j} x_i) = 0$ forces $x_i x_j = 0$) for Jordan orthogonality.

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LEMMA. If $x_1 \perp x_2$ and J has no nilpotent elements, then for $i \neq j$

- (i) $x_j^2 \perp x_i$,
- (ii) $U_{x_i}U_{x_j} = V_{x_i, x_j} = 0$ on J ,
- (iii) $x_i^\perp = \{y \in J \mid y \perp x_i\}$ is a linear subspace closed under squares,
- (iv) $y \perp x_1 + x_2 \Rightarrow y \perp x_1, y \perp x_2$.

PROOF. (i)

$$x_i \circ x_j^2 = (x_i \circ x_j) \circ x_j - 2U_{x_j}x_i \quad (\text{by MacDonald's principle}) = 0,$$

$$U_{x_j}^2x_i = U_{x_j}(U_{x_j}x_i) \quad (\text{MacDonald}) = 0,$$

$$\begin{aligned} (U_{x_i}x_j^2)^2 &= U_{x_i}\{U_{x_j}[(x_i \circ x_j)^2 - x_i \circ U_{x_j}x_i] - (U_{x_j}x_i)^2\} \quad (\text{MacDonald}) \\ &= 0 \end{aligned}$$

forces $U_{x_i}x_j^2 = 0$ by absence of nilpotents.

$$(ii) \quad (U_{x_i}U_{x_j}a)^2 = U_{x_i}U_{x_j}U_a(U_{x_j}x_i^2) = 0 \quad \text{by (i)}$$

(from $(U_x y)^2 = U_x U_y x^2$ by MacDonald),

$$\begin{aligned} (V_{x_i, x_j}a)^2 &= U_{x_i}U_{x_j}a^2 + U_aU_{x_j}x_i^2 + U_{x_i, a}U_{x_j}(x_i \circ a) - U_{x_i}x_j \circ U_a x_j \\ &\quad (\text{follows from linearizing } (U_x x_j)^2 = \dots) \\ &= 0 + 0 + U_{x_i, a}[\{x_j \circ x_i a x_j\} - x_i \circ U_{x_j}a] - 0 \quad (\text{MacDonald}) \\ &= -\{x_i x_i \circ U_{x_j}a a\} = -\{x_i^2(U_{x_j}a)a\} - U_{x_i}(U_{x_j}a) \circ a \quad (\text{MacDonald}) \\ &= -\{x_i^2 x_j U_a x_j\} - 0 \quad (\text{MacDonald}) \\ &= -\{x_i x_i \circ x_j(U_a x_j)\} + U_{x_i}x_j \circ (U_a x_j) \quad (\text{MacDonald}) \\ &= 0. \end{aligned}$$

(iii) If $y_1, y_2 \perp x$ then clearly $\Phi y_1 \perp x$, and $y_1 + y_2 \perp x$ since

$$\begin{aligned} (U_{y_1+y_2}x)^2 &= (U_{y_1, y_2}x)^2 \\ &= U_{y_1}(U_x y_2^2) + U_{y_2}(U_x y_1^2) + U_{y_1, y_2}U_x(y_1 \circ y_2) - (U_{y_1}x) \circ (U_{y_2}x) \\ &\quad (\text{linearizing } (U_y x)^2 = \dots) \\ &= U_{y_1, y_2}[\{x \circ y_1 y_2 x\} - y_1 \circ U_x y_2] \quad (\text{MacDonald}) \\ &= 0. \end{aligned}$$

(iv) We have $U_{x_i}y = 0$ since

$$U_{x_i}y = U_{(x_i+x_j)-x_j}y = U_{x_j}y \quad (\text{by } x_i + x_j \perp y \text{ and (ii)})$$

and

$$(U_{x_i}y)^3 = (U_{x_i}U_y U_{x_i})(U_{x_i}y) \quad (\text{MacDonald}) = U_{x_i}U_y(U_{x_i}U_{x_j}y) = 0 \quad \text{by (ii);}$$

then $U_{x_i}y^2 = 0$ since by (iii) $y^2 \perp x_i + x_j$, hence $U_y x_i = U_y x_i^2 = 0$ since $(U_y x_i)^2 = U_y(U_{x_i}y^2) = 0$ and $(U_y x_i^2)^2 = U_y U_{x_i}(U_{x_i}y^2) = 0$ (as we just saw), and finally $x_i \circ y = 0$ since

$$(x_i \circ y)^2 = y \circ U_{x_i}y + U_{x_i}y^2 + U_y x_i^2 \quad (\text{MacDonald}) = 0$$

by the previous.

2. Ordering.

DEFINITION. $x \leq y$ if $x - y \perp x$, i.e. if $y = x + x'$ for $x' \perp x$. By definition of orthogonality this means

$$(0i) \ x \circ (x - y) = 0,$$

$$(0ii) \ U_x(x - y) = 0,$$

(0iii) $U_{x-y}x = 0$ and in the presence of (0i), (0ii) the latter is equivalent to $U_x x = U_y x$ (since $U_{x-y}x + (U_x y - U_y x) = (V_x^2 - 2U_x)(x - y)$ (MacDonald) = 0), so $x - y \perp x$ iff

$$(0i) \ V_x x = V_x y,$$

$$(0ii) \ U_x x = U_x y,$$

(0iii) $U_x x = U_y x$. This agrees with our definition in (5) for linear Jordan algebras where $\frac{1}{2} \in \Phi$,

$$(Li) \ x \cdot x = x \cdot y,$$

$$(Lii) \ x^2 \cdot x = x^2 \cdot y,$$

(Liii) $x \cdot x^2 = x \cdot y^2$, since $a \circ b = 2a \cdot b$, $U_a b = 2a \cdot (a \cdot b) - a^2 \cdot b$. For $J = A^+$ with A an associative algebra without nilpotent elements this ordering agrees with the Abian ordering in the associative algebra A

$$x \leq y \Leftrightarrow xy = yx = x^2$$

(\Leftarrow is clear, for \Rightarrow note $x - y \perp x \Rightarrow (x - y)x = x(x - y) = 0$ (by our remark after the definition of orthogonality)). Note

$$(\wedge) \quad x \leq y \Rightarrow y^\perp \subset x^\perp \text{ (since } z \perp y = x + x' \Rightarrow z \perp x \text{ by Lemma (iv))}.$$

THEOREM 1. *The relation \leq is a partial ordering for a quadratic Jordan algebra J if and only if J has no nilpotent elements.*

PROOF. If J has nilpotent elements it has $x \neq 0$ with $x^2 = 0$, so $0 \leq x \leq 0$ and \leq is not antisymmetric. Now assume that J has no nilpotent elements (so the Lemma applies). Clearly \leq is reflexive ($0 \perp x$); it is antisymmetric since $x \leq y \leq x \Rightarrow x - y \perp x, y \Rightarrow x - y \perp x - y$ (by Lemma (iii)) $\Rightarrow x - y = 0$; it is transitive since $x \leq y \leq z \Rightarrow x - y \in x^\perp, y - z \in y^\perp \subset x^\perp$ (by (\wedge)), so $x - z = (x - y) + (y - z) \in x^\perp$ (by Lemma (iii)) and $x \leq z$. Note in particular that our hypothesis (P), $(x, x, y) = 0 \Rightarrow (xy, x, y) = 0$, of (5) is superfluous, answering the question raised on [5, p. 383].

We remark that if $x \leq y$ then $p(x, y) = p(x, x)$ for all Jordan polynomials in x and y all of whose monomials have degree at least one in both x and y . We also remark that if J has no nilpotents then $x_1 \perp x_2$ iff $U_{x_1}U_{x_2} = 0$ on J (\Rightarrow from Lemma (ii), \Leftarrow since $U_{x_i}x_j = 0$ (since $(U_{x_i}x_j)^2 = U_{x_i}U_{x_j}U_{x_i}1 = 0$), similarly $U_{x_i}x_j^2 = 0$, and so $x_1 \circ x_2 = 0$ (since $(x_1 \circ x_2)^2 = x_1 \circ U_{x_2}x_1 + U_{x_2}x_1^2 + U_{x_1}x_2^2 = 0$).

3. A structure theorem.

DEFINITION. J is *orthogonally complete* if every family $\{x_\alpha\}$ of pairwise orthogonal elements has a supremum relative to the ordering \leq .

DEFINITION. An element $e \in J$ is a *hyperatom* if it is a central division idempotent (so has Peirce decomposition $J = J + 1 \oplus J_0$ for ideal J_i with $J_1 = U_e J$ a Jordan division algebra (in the sense that all operators U_{x_1} are surjective on J_1 , hence bijective there)).

DEFINITION. J is *hyperatomic* if for every $a \neq 0$ in J there is a hyperatom e with $U_e a \neq 0$.

STRUCTURE THEOREM. *J is isomorphic to a direct product of Jordan division algebras if and only if J has no nilpotent elements, is hyperatomic and orthogonally complete with respect to \leq .*

PROOF. Suppose first that $J \cong \pi J_i$ for division algebras J_i with units e_i . Clearly J has no nilpotent elements, and is hyperatomic since any $x \neq 0$ has some $x_i = U_{e_i} x \neq 0$ where the e_i are hyperatoms. To see that J is orthogonally complete, if $x = \pi x_i$, $y = \pi y_i$ are orthogonal then $x_i \neq 0$ implies $y_i = 0$ (since $U_x y = 0$ by orthogonality forces $U_{x_i} y_i = 0$ in the division algebra J_i), so x and y have disjoint support (support being the set of indices i for which $x_i \neq 0$). If $\{x_\alpha\}$ is an orthogonal family then the support sets $S_\alpha = \text{support}(x_\alpha)$ are disjoint, and $x = \pi x_i$ is the supremum of the x_α , where $x_i = x_{\alpha i}$ if the index i lies in a (unique) S_α and $x_i = 0$ if i lies in no S_α .

Conversely, suppose J is hyperatomic with hyperatoms $\{e_i\}$. Then the Peirce decomposition relative to e_i is $J = J_1(e_i) \oplus J_0(e_i)$, and U_{e_i} is an algebra homomorphism of J onto the division algebra $J_i = U_{e_i} J$, and $\theta(x) = \pi(U_{e_i} x)$ is an algebra homomorphism of J into πJ_i . This is injective since $\theta(x) = 0$ implies $U_{e_i} x = 0$ for each i , hence $x = 0$ by definition of hyperatomic. If J has no nilpotent elements \leq is a partial order, and if J is orthogonally complete θ is surjective: given any family $x_i \in J_i$ their supremum x must be the element such that $\theta(x) = \pi x_i$, since $U_{e_j} x = x_j$ for each j : $U_{e_j} x = \sup U_{e_j} x_i = U_{e_j} x_j = x_j$ ($U_{e_j} x_i = U_{e_j} U_{e_i} x_i = 0$ for $i \neq j$ by Lemma (ii) since $e_i \perp e_j$).

REFERENCES

1. A. Abian, *Direct product decomposition of commutative semisimple rings*, Proc. Amer. Math. Soc. **24** (1970), 502–507.
2. —, *Order in a special class of rings and a structure theorem*, Proc. Amer. Math. Soc. **52** (1975), 45–49.
3. —, *Addendum to “Order in a special class of rings and a structure theorem”*, Proc. Amer. Math. Soc. **61** (1976), 188.
4. M. Chacron, *Direct product of division rings and a paper of Abian*, Proc. Amer. Math. Soc. **29** (1971), 259–262.
5. S. González and C. Martínez, *Order relation in Jordan rings and a structure theorem*, Proc. Amer. Math. Soc. **98** (1986), 379–388.
6. —, *Order relation in JB-algebras*, Comm. Algebra **15** (1987), 1869–1877.
7. N. Jacobson, *Lectures on quadratic Jordan algebras*, Lecture Notes, Tata Institute, Bombay, 1969.
8. —, *Structure theory of Jordan algebras*, Lecture Notes in Math., University of Arkansas, 1971.
9. O. Loos, *Jordan pairs*, Lecture Notes in Math., vol. 460, Springer-Verlag, Berlin and New York, 1975.
10. K. McCrimmon, *A general theory of Jordan rings*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 1072–1079.
11. H. C. Myung and L. Jimenez, *Direct product decomposition of alternative rings*, Proc. Amer. Math. Soc. **47** (1975), 53–60.

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