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THE REGULARITY OF DUNFORD-PETTIS OPERATORS

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ABSTRACT. Let λ denote a symmetric, solid Banach sequence space having $\{e_i\}_{i=1}^{\infty}$ as a symmetric basis and considered as a Banach lattice with order defined coordinatewise. A complete description of the relationship between regular and Dunford-Pettis operators $T\colon L^1[0,1] \to \lambda$ is given. The results obtained complete earlier work of Gretsky and Ostroy and of the author in this area.

1. Introduction. If X and Y are Banach spaces, a bounded linear operator $T\colon X \to Y$ is called a *Dunford-Pettis operator* (or, as we write, a D-P operator) if T maps weakly convergent sequences in X to norm convergent ones in Y. In the case where X and Y are Hilbert spaces such operators were once called "completely continuous", but the terminology gradually disappeared in favor of the notion of a compact operator (with which it agrees when the domain of the operator is a reflexive Banach space). In general, however, compactness of an operator is a more restrictive condition, a fact which is rather dramatically apparent in the case of operators defined on l^1, all of which are D-P. The term "Dunford-Pettis operator" was introduced by Grothendieck [**8**] in view of the pioneering work of Dunford and Pettis [**3**] in investigating the properties of such operators on L^1 and $C(S)$-spaces. Due to its importance for various applications subsequent work has continued to focus on understanding the structure of D-P operators from the space $L^1[0,1]$ to various specific, as well as general, Banach spaces (e.g. [**1, 2, 5, 6, 7, 9, 11, 13, 14, 16**]).

A particular case in point is the paper of Gretsky and Ostroy [**6**] on D-P operators from $L^1[0,1]$ to certain Banach lattices. Motivated by considerations of mathematical economics [**5**] they showed that every regular operator (i.e. a difference of positive operators) from $L^1[0,1]$ to a Banach lattice having an "order compatible" Schauder basis is a D-P operator; in particular (their case of greatest interest) every regular operator from $L^1[0,1]$ to c_0 is D-P. In a recent paper [**9**] the author showed the converse of this last result is also true, so the D-P operators from $L^1[0,1]$ to c_0 are precisely the regular operators.

The question which now arises is whether the converse of the Gretsky-Ostroy theorem is true in general. In particular, suppose λ is a symmetric, solid, Banach sequence space (i.e., if $\{a_i\}_{i=1}^{\infty} \in \lambda$ then $\{a_{\pi(i)}\}_{i=1}^{\infty} \in \lambda$ for any permutation π, and if $|b_i| \le |a_i|$ for all i then $(b_i) \in \lambda$). Suppose, too, that the unit vectors $\{e_j\}_{j=1}^{\infty}$ in λ, defined by $e_j = \{\delta_{i,j}\}_{i=1}^{\infty}$, form a symmetric basis for λ and that the order

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on λ is defined by $\{a_i\}_{i=1}^{\infty} \leq \{b_i\}_{i=1}^{\infty} \Leftrightarrow a_i \leq b_i$ for all i. Then $\{e_j\}_{j=1}^{\infty}$ is an order compatible basis for λ, so every regular operator $T\colon L^1[0,1] \to \lambda$ is D-P. The question we consider is the converse: *Is every* D-P *operator* $T\colon L^1[0,1] \to \lambda$ *a regular operator*?

The purpose of the present paper is to give a complete solution to this problem. The interesting aspect of the solution itself is the fact that the positive answer obtained for the case $\lambda = c_0$ mentioned above is completely atypical, in the following sense: If $\lambda = \lambda^{xx}$ then *every* bounded linear operator $T\colon L^1[0,1] \to \lambda$ is regular, while if $\lambda \neq \lambda^{xx}$ (the case of c_0) then the *only* time every D-P operator $T\colon L^1[0,1] \to \lambda$ is regular is when $\lambda = c_0$; hence the result proved in [**9**] is more fortuitous than characteristic. Finally, we show that for any λ every weakly compact operator $T\colon L^1[0,1] \to \lambda$ is regular, a satisfying result pertinent to this questions since every such weakly compact operator is a D-P operator [**10**, p. 182].

2. If E and F are Banach spaces we denote the set of all bounded linear operators from E to F by $\mathscr{L}(E,F)$. If E is a Banach space and $x \in E$, we denote the norm of x in E by $\|x\|_E$. In the case where E is L^p or l^p for $1 \leq p \leq +\infty$, we will write $\|x\|_E = \|x\|_p$. *Throughout the paper λ will always denote a symmetric, solid, Banach sequence space in which the unit vectors $\{e_j\}_{j=1}^{\infty}$ form a symmetric Schauder basis and in which the order is defined coordinatewise* (as we outlined in §1). In particular, then, $\{e_j\}_{j=1}^{\infty}$ is an order compatible basis for λ, so every regular operator $T\colon L^1[0,1] \to \lambda$ is a D-P operator.

Recall that the *Köthe dual* of λ is the sequence space

$$\lambda^x = \left\{ \{b_i\}_{i=1}^{\infty} \,\middle|\, \sum_{i=1}^{\infty} |a_i||b_i| < +\infty \text{ for all } \{a_i\}_{i=1}^{\infty} \text{ in } \lambda \right\}.$$

By the Uniform Boundedness Principle and the fact that $\{e_i\}_{i=1}^{\infty}$ is a basis for λ it follows that λ^x may be identified with the dual space λ^* of λ. Hence we consider λ^x as a Banach space with norm defined by

$$\|\{b_i\}_{i=1}^{\infty}\|_{\lambda^x} = \sup\left\{ \sum_{i=1}^{\infty} |a_i||b_i| \,\middle|\, \|\{a_i\}_{i=1}^{\infty}\|_{\lambda} \leq 1 \right\}.$$

Note that unless $\{e_i\}_{i=1}^{\infty}$ is a basis for λ^x we will *not* generally have $\lambda^{xx} = \lambda^{**}$, although we always have $\lambda \subset \lambda^{xx}$. Those space λ for which $\lambda^{xx} = \lambda$ are called *perfect.* (For more detailed information on such matters see §30 of the book *Topological vector spaces.* I, by G. Köthe, Springer-Verlag, 1969.)

We begin with the following observation about operators from $L^1[0,1]$ to λ which will be seen to lead naturally to our main results.

THEOREM 1. *If $T\colon L^1[0,1] \to \lambda$ is any bounded linear operator, then*

$$\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda^{xx}.$$

If $\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda$, then T is a D-P operator.

PROOF. Since $T \in \mathscr{L}(L^1[0,1], \lambda)$, $T^*\colon \lambda^x \to L^\infty[0,1]$ is both norm and weak*-continuous. If $\{a_i\}_{i=1}^{\infty} \in \lambda^x$ with $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$ then $\sum_{i=1}^{\infty} \varepsilon_i a_i e_i$ is weak* convergent to $\{\varepsilon_i a_i\}_{i=1}^{\infty}$ for every sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ with $|\varepsilon_i| = 1$, and $\|\sum_{i=1}^{\infty} \varepsilon_i a_i e_i\|_{\lambda^x} \leq 1$

also. Therefore $\sum_{i=1}^{\infty} \varepsilon_i a_i T^* e_i$ is weak*-convergent in $L^\infty[0,1]$ and

$$\left\| \sum_{i=1}^{\infty} \varepsilon_i a_i T^* e_i \right\|_\infty \leq \|T^*\|.$$

From the definition of the norm in $L^\infty[0,1]$ it follows that $\sum_{i=1}^{\infty} |a_i||T^* e_i(t)| \leq \|T^*\|$ a.e. in $[0,1]$, so by the Dominated Convergence Theorem the series $\sum_{i=1}^{\infty} |a_i||T^* e_i|$ converges in $L^1[0,1]$ and $\| \sum_{i=1}^{\infty} |a_i||T^* e_i| \|_1 \leq \|T^*\|$. But then $\sum_{i=1}^{\infty} |a_i| \|T^* e_i\|_1 \leq \|T^*\|$ for all $(a_i) \in \lambda^x$ with $\|(a_i)\|_{\lambda^x} \leq 1$, and by definition it follows that

$$\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda^{xx}.$$

Now suppose $\{\|T^* e_i\|_1\}_{i=1}^{\infty}$ is actually in $\lambda \subset \lambda^{xx}$. It is well known that $T\colon L^1[0,1] \to \lambda$ is a D-P operator if and only if $T \circ i\colon L^\infty[0,1] \to L^1[0,1] \to \lambda$ is compact [**1**] (where $i\colon L^\infty[0,1] \to L^1[0,1]$ is the canonical injection map). Since $\{\|T^* e_i\|_1\} \in \lambda$ and $\{e_i\}_{i=1}^{\infty}$ is a basis for λ, $\| \sum_{i=n}^{\infty} \|T^* e_i\|_1 e_i \|_\lambda \to 0$ as $n \to \infty$. Therefore $| \sum_{i=n}^{\infty} a_i \|T^* e_i\|_1 | \to 0$ uniformly over $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$, so given $\varepsilon > 0$ choose n so that $| \sum_{i=n}^{\infty} a_i \|T^* e_i\|_1 | < \varepsilon$ for all $\|(a_i)\|_{\lambda^x} \leq 1$. Then $| \sum_{i=n}^{\infty} a_i \langle T^* e_i, g\rangle | < \varepsilon$ whenever $\|(a_i)\|_{\lambda^x} \leq 1$ and $\|g\|_\infty \leq 1$, or $|\langle \sum_{i=n}^{\infty} \langle T \circ i(g), e_i\rangle e_i, \{a_i\}_{i=1}^{\infty}\rangle| < \varepsilon$ whenever $\|(a_i)\|_{\lambda^x} \leq 1$ and $\|g\|_\infty \leq 1$. That is, $\| \sum_{i=n}^{\infty} \langle T \circ i(g), e_i\rangle e_i \|_\lambda \to 0$ as $n \to \infty$, uniformly over g in the unit ball of $L^\infty[0,1]$, so $\{T \circ ig \mid \|g\|_\infty \leq 1\}$ is conditionally compact in λ [**4**, p. 260]. Therefore $T \circ i$ is compact and it follows that T is a D-P operator.

It is natural to ask whether the converse of the last assertion of Theorem 1 is true, and hence whether it is possible to characterize the D-P operators from $L^1[0,1]$ to λ by the condition that $\{\|T^* e_i\|\}_{i=1}^{\infty} \in \lambda$. According to the first part of Theorem 1 this will be true whenever $\lambda = \lambda^{xx}$, and it was shown in [**9**] to also be true when $\lambda = c_0$. It turns out, however, that this is not always the case. In fact, (as in the case $\lambda = c_0$ in [**9**]) the condition $\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda$ actually characterizes the regularity of operators from $L^1[0,1]$ to λ.

THEOREM 2. *$T\colon L^1[0,1] \to \lambda$ is regular $\Leftrightarrow$ $\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda$.*

PROOF. ($\Rightarrow$) To show that a regular operator $T\colon L^1[0,1] \to \lambda$ has the property that $\{\|T^* e_i\|_1\|\} \in \lambda$, we need only show this is true whenever $T \geq 0$. Therefore, suppose $T\colon L^1[0,1] \to \lambda$ is a positive operator. Then T is a D-P operator [**6**], so $T \circ i\colon L^\infty[0,1] \to L^1[0,1] \to \lambda$ is compact [**1**]. We showed in Theorem 1 that then $| \sum_{i=n}^{\infty} \langle T^* e_i, g\rangle a_i | \to 0$ as $n \to \infty$, uniformly over $g \in L^\infty[0,1]$ with $\|g\|_\infty \leq 1$ and $\{a_i\}_{i=1}^{\infty} \in \lambda^x$ with $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$, and since $T \geq 0$ we have $T^* e_i \geq 0$ a.e. in $[0,1]$ for all $i = 1, 2, \ldots$. Given any $\varepsilon > 0$ choose N so that if $n \geq N$ then $| \sum_{i=n}^{\infty} \langle T^* e_i, |g|\rangle |a_i|| < \varepsilon$ whenever $\|g\|_\infty \leq 1$ and $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$. That is, $|\langle \sum_{i=n}^{\infty} |a_i| T^* e_i, |g|\rangle| < \varepsilon$ for all $\|g\|_\infty \leq 1$ and all $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$, so $\| \sum_{i=n}^{\infty} |a_i| T^* e_i \|_1 < \varepsilon$ whenever $n \geq N$ and $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$. But since $T^* e_i \geq 0$ a.e. this says that $\sum_{i=n}^{\infty} |a_i| \|T^* e_i\|_1 < \varepsilon$ whenever $\|(a_i)\|_{\lambda^x} \leq 1$ and $n \geq N$, so $\sum_{i=1}^{\infty} \|T^* e_i\|_1 e_i$ converges in λ; that is, $\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda$.

($\Leftarrow$) On the other hand, suppose $T\colon L^1[0,1] \to \lambda$ and $\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda$. To show T is regular we first show that we can define a bounded linear operator $|T|\colon L^1[0,1] \to \lambda$ by $|T|(f) = \{\langle f, |T^* e_i|\rangle\}_{i=1}^{\infty}$. To do this, let $\varepsilon > 0$ be given, let $f \in L^1[0,1]$, and choose $g \in L^\infty[0,1]$ so that $\|f - g\|_1 < \varepsilon / 2\|T\|$. Then for any

$n \geq 1$ and any $\{a_i\}_{i=1}^{\infty} \in \lambda^x$ we have

$$\left|\left|\sum_{i=n}^{\infty} a_i \langle f, |T^* e_i| \rangle\right|\right| \leq \left|\sum_{i=n}^{\infty} |a_i| |\langle f - g, |T^* e_i| \rangle|\right| + \left|\sum_{i=n}^{\infty} |a_i| \langle g, |T^* e_i| \rangle\right|.$$

Now $\sum_{i=1}^{\infty} a_i T^* e_i$ is weak*-convergent in $L^\infty[0,1]$ for every $\{a_i\}_{i=1}^{\infty} \in \lambda^x$ and $\|\sum_{i=1}^{\infty} a_i T^* e_i\|_\infty \leq \|T^*\|$ whenever $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$. That is,

$$\operatorname{ess\,sup} \left|\sum_{i=1}^{\infty} \varepsilon_i a_i T^* e_i(t)\right| \leq \|T^*\|$$

for all $\|\{a_i\}\|_{\lambda^x} \leq 1$ and for all $\{\varepsilon_i\}$ with $|\varepsilon_i| = 1$; it follows that $\| \sum_{i=1}^{\infty} |a_i| |T^* e_i| \|_\infty \leq \|T^*\| = \|T\|$ for all such $\{a_i\} \in \lambda^x$, and hence that $|\sum_{i=n}^{\infty} |a_i| \langle |T^* e_i|, f - g \rangle| \leq \|f - g\|_1 \|T\| < \varepsilon/2$ whenever $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$, and for all n. Moreover, since $\{\|T^* e_i\|_1\}_{i=1}^{\infty} \in \lambda$ there is an integer N such that if $n \geq N$ then $\| \sum_{i=n}^{\infty} \|T^* e_i\| e_i \|_\lambda < \varepsilon / 2\|g\|_\infty$. Therefore if $n \geq N$ and $\|\{a_i\}\|_{\lambda^x} \leq 1$ it follows that

$$\left|\sum_{i=n}^{\infty} |a_i| \langle |T^* e_i|, g \rangle\right| \leq \|g\|_\infty \sum_{i=n}^{\infty} |a_i| \|T^* e_i\|_1 < \|g\|_\infty \frac{\varepsilon}{2\|g\|_\infty} = \frac{\varepsilon}{2},$$

so if $n \geq N$ we have, by the above, that $|\sum_{i=n}^{\infty} a_i \langle f, |T^* e_i| \rangle| < \varepsilon$ for all $\|\{a_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$. Hence the sequence $\{\langle |T^* e_i|, f \rangle\}_{i=1}^{\infty}$ is in λ for all $f \in L^1[0,1]$, so the operator $|T| \colon L^1[0,1] \to \lambda$ for which $|T|(f) = \{\langle |T^* e_i, f \rangle\}_{i=1}^{\infty}$ is well defined and bounded.

Now clearly $|T| \geq T$ and $|T| \geq -T$ (since if $f \geq 0$ in $L^1[0,1]$ then

$$|\langle Tf, e_i \rangle| = |\langle T^* e_i, f \rangle| = \left|\int_0^1 T^* e_i(t) f(t)\, dt\right| \leq \int_0^1 |T^* e_i|(t) f(t)\, dt = \langle |T^* e_i|, f \rangle$$

for all i), so we have $T = ((|T| + T)/2) - ((|T| - T)/2)$, a difference of positive operators, and T is regular.

COROLLARY. *If $T \colon L^1[0,1] \to \lambda$ is a bounded linear operator and $\lambda^{xx} = \lambda$, then T is regular and a D-P operator.*

We see in Corollary 1 the essential simplicity of the case $\lambda = \lambda^{xx}$, while if $\lambda \neq \lambda^{xx}$ the relationship between regular operators and D-P operators becomes more interesting. In all cases regularity implies the D-P property [**6**], and when $\lambda = c_0$ it is known that the converse also holds [**9**]. Surprisingly, this turns out to be the *only* such case (i.e. where $\lambda \neq \lambda^{xx}$) for which this is true.

THEOREM 3. *If $\lambda \neq \lambda^{xx}$ then every D-P operator $T \colon L^1[0,1] \to \lambda$ is a regular operator $\Leftrightarrow \lambda = c_0$.*

PROOF. As we have already mentioned, if $\lambda = c_0$ then every D-P operator is regular.

Suppose, then, that $\lambda \neq \lambda^{xx}$ and that $\lambda \neq c_0$. To show there is a D-P operator which is not regular we need only demonstrate (by Theorem 2) a D-P operator $T \colon L^1[0,1] \to \lambda$ for which $\{\|T^* e_i\|_1\}_{i=1}^{\infty} \notin \lambda$.

We first note that since $\lambda \neq c_0$ it follows that $\lambda^{xx} \neq l^\infty$. Otherwise, if $\{c_i\}_{i=1}^{\infty} \in l^\infty$ then $\{c_i\}_{i=1}^{\infty} \in \lambda^{xx}$ so $\sum_{i=1}^{\infty} |c_i| |b_i| \leq \|\{c_i\}_{i=1}^{\infty}\|_{\lambda^{xx}}$ whenever $\|\{b_i\}_{i=1}^{\infty}\|_{\lambda^x} \leq 1$; in particular this would be true when $c_i = 1$ for all i, implying $\{b_i\}_{i=1}^{\infty} \in l^1$ for all $\{b_i\}_{i=1}^\infty \in \lambda^x$ and hence that $\lambda^x = l^1$ (since $l^1 \subset \lambda^x$ in all cases). But then whenever $\{c_i\}_{i=1}^\infty \in c_0$ and $p < q$

$$\left\|\sum_{i=p}^{q} c_i e_i\right\|_\lambda = \sup\left\{\sum_{i=p}^{q}|c_i||b_i| \;\middle|\; \|\{b_i\}_{i=1}^\infty\|_{\lambda^x} \leq 1\right\}$$
$$\leq \sup\left\{\sum_{i=p}^{q}|c_i||b_i| \;\middle|\; \|\{b_i\}_{i=1}^\infty\|_1 \leq K\right\}$$

for some constant k, (since λ^x is isomorphic to l^1 by the above). Since this last is $\leq K\sup_{i\geq p}|c_i| \to 0$ as $p \to \infty$ it follows that $\{c_i\}_{i=1}^\infty \in \lambda$ and since $\lambda \subset c_0$ (always) we have $\lambda = c_0$, a contradiction. Therefore, it must be that $\lambda^{xx} \neq l^\infty$. From this we also see that λ^{xx} *must be a subset of* c_0. If not, there is a sequence $\{a_i\}_{i=1}^\infty$ in λ^{xx} but not in c_0, and hence an $\varepsilon > 0$ and a subsequence $\{a_{i_k}\}_{k=1}^\infty$ of $\{a_i\}_{i=1}^\infty$ for which $|a_{i_k}| \geq \varepsilon$ for all k. Since λ^{xx} is both symmetric and solid the sequence $\{|a_{i_1}|, |a_{i_2}|, \dots\}$ is also in λ^{xx} and hence so is any bounded sequence $\{b_j\}_{j=1}^\infty$ since we can write $\{b_j\}_{j=1}^\infty$ as the sequence

$$\left\{\frac{b_1}{|a_{i_1}|}a_{i_1}, \frac{b_2}{|a_{i_2}|}a_{i_2}, \dots, \frac{b_k}{|a_{i_k}|}a_{i_k}, \dots\right\}, \quad \text{where } \sup_k \frac{|b_k|}{|a_{i_k}|} \leq \frac{1}{\varepsilon}\sup_k |b_k| < +\infty.$$

That is, $l^\infty \subset \lambda^{xx}$ and since the reverse inclusion always holds we would have $l^\infty = \lambda^{xx}$, a contradiction to the above result.

Now let $\{r_n\}_{n=1}^\infty$ denote the set of Rademacher functions on $[0,1]$ [**15**, p. 396]. It is well known that these functions have the property that $\|r_n\|_\infty = \|r_n\|_1 = 1$, $n = 1,2,3,\dots$, and $\{r_n\}_{n=1}^\infty$ is an orthonormal sequence in $L^2[0,1]$. Choose any sequence $\{a_n\}_{n=1}^\infty$ in λ^{xx} but not in λ, so as we noted above $\{a_n\}_{n=1}^\infty \in c_0$. For any function $g \in L^2[0,1]$ the sequence $\{\langle r_n, g\rangle\}_{n=1}^\infty \in l^2 \subset c_0$, so since $\|r_n\|_\infty = 1$ for all n it follows by a standard approximation argument that for any $f \in L^1[0,1]$ the sequence $\{\langle r_n, f\rangle\}_{n=1}^\infty \in c_0$. Hence the sequence $\{a_n\langle r_n, f\rangle\}_{n=1}^\infty \in \lambda$ for all $f \in L^1[0,1]$, since

$$\left\|\sum_{n=p}^{q} a_n\langle r_n, f\rangle e_n\right\|_\lambda = \sup\left\{\sum_{n=p}^{q}|a_n||\langle r_n, f\rangle||b_n| \;\middle|\; \|\{b_n\}_{n=1}^\infty\|_{\lambda^x} \leq 1\right\}$$
$$\leq \sup_{n\geq p}|\langle r_n, f\rangle| \sup\left\{\sum_{n=p}^{q}|a_n||b_n| \;\middle|\; \|\{b_n\}_{n=1}^\infty\|_{\lambda^x} \leq 1\right\}$$
$$\leq \sup_{n\geq p}|\langle r_n, f\rangle| \cdot \|\{a_n\}_{n=1}^\infty\|_{\lambda^{xx}} \to 0 \text{ as } p \to \infty,$$

and so we can define a bounded linear operator $T\colon L^1[0,1] \to \lambda$ by

$$Tf = \sum_{n=1}^{\infty} a_n\langle r_n, f\rangle e_n.$$

Clearly $T^*e_n = a_n r_n$ for n, so $\{\|T^*e_n\|_1\}_{n=1}^\infty = \{|a_n|\}_{n=1}^\infty \notin \lambda$, and by Theorem 2 it follows that T is not regular. To see that T is a D-P operator, recall that if $i\colon L^2[0,1] \to L^1[0,1]$ is the injection map, then T is a D-P operator $\Leftrightarrow$ $T \circ i\colon L^2[0,1] \to L^1[0,1] \to \lambda$ is compact [**1**], and this last is true $\Leftrightarrow$

$\sum_{n=1}^{\infty}\langle T\circ i(f), e_n\rangle e_n$ converges in λ uniformly over $f\in L^2[0,1]$ with $\|f\|_2\le 1$ [**4**, p. 260].

If $f\in L^2[0,1]$ and $\|f\|_2\le 1$, then for any N we have

$$\left\|\sum_{n=N}^{\infty}\langle T\circ i(f), e_n\rangle e_n\right\|_\lambda = \left\|\sum_{n=N}^{\infty} a_n\langle r_n, f\rangle e_n\right\|_\lambda = \sup_{\|\{b_n\}\|_{\lambda^x}\le 1}\left|\sum_{n=N}^{\infty}|a_n||b_n||\langle r_n, f\rangle|\right|.$$

Since $\{\{\langle r_n, f\rangle\}_{n=1}^{\infty} \mid \|f\|_2\le 1\}$ is precisely the unit ball in l^2 (recall $\{r_n\}$ is an O.N. sequence in $L^2[0,1]$) it follows that

$$\begin{aligned}\sup_{\|f\|_2\le 1}\left\|\sum_{n=N}^{\infty} a_n\langle r_n, f\rangle e_n\right\|_\lambda &\le \sup_{\|\{b_n\}\|_{\lambda^x}\le 1}\left\|\sum_{n=N}^{\infty} a_n b_n e_n\right\|_2\\ &\le \left[\sup_{n\ge N}|a_n||b_n|\right]^{1/2}\sup_{\|\{b_n\}\|_{\lambda^x}\le 1}\left[\sum_{n=N}^{\infty}|a_n||b_n|\right]^{1/2}\\ &\le \sup_{n\ge N}|a_n|^{1/2}\|\{a_n\}\|_\lambda^{1/2}\end{aligned}$$

where this goes to zero as $N\to\infty$ since $\{a_n\}\in c_0$. Hence $\|\sum_{n=N}^{\infty} a_n\langle r_n, f\rangle e_n\|_\lambda\to 0$ uniformly over $\|f\|_2\le 1$ and, as we remarked above, it follows that T is a D-P operator which is not regular.

The results given here (along with the Gretsky-Ostroy Theorem) provide a complete description of the relationship between D-P operators and regular operators from $L^1[0,1]$ to λ; in particular, when $\lambda\ne\lambda^{xx}$ and $\lambda\ne c_0$ there always exist D-P operators from $L^1[0,1]$ to λ which fail to be regular. In contrast we now show that in all cases every weakly compact operator is regular. The significance of this result in the context of those obtained previously lies in the fact that every weakly compact operator $T\colon L^1[0,1]\to\lambda$ is a D-P operator [**10**, p. 182], thereby illustrating the fundamental difference between weakly compact and D-P operators in the case where $\lambda\ne\lambda^{xx}$.

THEOREM 4. *Every weakly compact operator from $L^1[0,1]$ to λ is regular.*

PROOF. Let $T\colon L^1[0,1]\to\lambda$ be a weakly compact operator. According to Theorem 2, T is regular if and only if $\{\|T^*e_i\|_1\}_{i=1}^{\infty}\in\lambda$, while by Theorem 1 $\{\|T^*e_i\|_1\}_{i=1}^{\infty}\in\lambda^{xx}$ in any case.

Suppose $\{\|T^*e_i\|_1\}_{i=1}^{\infty}\in\lambda^{xx}$, but is not in λ. Then $\sum_{i=1}^{\infty} b_i\|T^*e_i\|_1$ converges pointwise, but not uniformly, over vectors $\{b_i\}_{i=1}^{\infty}$ is the unit ball of λ^x. Hence there is some $\varepsilon>0$ and an increasing sequence $\{p_k\}_{k=1}^{\infty}$ of integers for which $\|\sum_{i=p_k+1}^{p_{k+1}}\|T^*e_i\|_1 e_i\|_\lambda\ge\varepsilon$, $k=1,2,\dots$; correspondingly, there is a sequence of unit vectors $\{b_i^{(k)}\}_{i=1}^{\infty}$, $k=1,2,\dots$, in λ^x for which $\sum_{i=p_k+1}^{p_{k+1}}|b_i^{(k)}|\|T^*e_i\|_1\ge\varepsilon$ for all k.

It is well known that the injection mapping $i\colon L^\infty[0,1]\to L^1[0,1]$ is weakly compact and hence a D-P operator [**10**, p. 182], so if $\{\sum_{i=p_k+1}^{p_{k+1}}|b_i^{(k)}||T^*e_i|\}_{k=1}^{\infty}$ were weakly convergent to zero in $L^\infty[0,1]$ then it would converge to zero in $L^1[0,1]$; but this would say $\{\sum_{i=p_k+1}^{p_{k+1}}|b_i^{(k)}|\|T^*e_i\|_1\}_{k=1}^{\infty}\to 0$, a contradiction to the above. Hence it must be that $\{\sum_{i=p_k+1}^{p_{k+1}}|b_i||T^*e_i|\}_{k=1}^{\infty}$ is *not* weakly convergent to zero in $L^\infty[0,1]$. Now $L^\infty[0,1]$ is order-isomorphic to a space $C(S)$ of continuous functions on some compact Hausdorff space S [**4**, p. 445]. If, under this isometry, T^*e_i corresponds to $f_i \in C(S)$, then $|T^*e_i|$ corresponds to $|f_i|$ and by the above the sequence $\{\sum_{i=p_k+1}^{p_{k+1}} |b_i^{(k)}||f_i|\}$ does not converge weakly to zero in $C(S)$. Consequently there is some point $s_0 \in S$ so that $\{\sum_{i=p_k+1}^{p_{k+1}} |b_i^{(k)}||f_i(s_0)|\}$ does not converge to zero [**4**, p. 265]. Choosing numbers $\varepsilon_i^{(k)}$ so that $\varepsilon_i^{(k)} b_i^{(k)} f_i(s_0) = |b_i^{(k)}||f_i(s_0)|$ for all i and k we then see that $\{\sum_{i=p_k+1}^{p_{k+1}} \varepsilon_i^{(k)} b_i^{(k)} f_i\}_{k=1}^{\infty}$ does not converge weakly to zero in $C(S)$. However the sequence $\{\sum_{i=p_k+1}^{p_{k+1}} \varepsilon_i^{(k)} b_i^{(k)} e_i\}_{k=1}^{\infty}$ is weak*-convergent to zero in λ^x, so since T^* is weakly compact the sequence $\{\sum_{i=p_k+1}^{p_{k+1}} \varepsilon_i^{(k)} b_i^{(k)} T^* e_i\}_{k=1}^{\infty}$ must converge weakly to zero in $L^\infty[0,1]$, implying $\{\sum_{i=p_k+1}^{p_{k+1}} \varepsilon_i^{(k)} b_i^{(k)} f_i\}_{k=1}^{\infty}$ converges weakly to zero in $C(S)$, and we reach a contradiction. Therefore it must be that $\sum_{i=1}^{\infty} b_i \|T^*e_i\|_1$ converges uniformly over $\|\{b_i\}\|_{\lambda^x} < 1$, hence that $\{\|T^*e_i\|_1\}_{i=1}^{\infty} \in \lambda$, and T is regular.

REMARK. The converse of Theorem 4 is, in general, not true. For example, if $f_n = \chi_{[0,1/n]}$ for $n = 1, 2, \ldots$ then $\{f_n\}_{n=1}^{\infty}$ is weak*-convergent to zero in $L^\infty[0,1]$ so $T = \sum_{n=1}^{\infty} f_n \otimes e_n$ is a positive linear operator from $L^1[0,1]$ to c_0, yet since $\{f_n\}$ does not converge weakly to zero in $L^\infty[0,1]$ (any convex combination of the functions f_n has norm 1 in $L^\infty[0,1]$) we see that T^*, hence T, is not weakly compact.

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