

## EXPONENTIAL DICHOTOMIES AND FREDHOLM OPERATORS

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ABSTRACT. It is shown that if the operator  $(Lx)(t) = \dot{x}(t) - A(t)x(t)$  is semi-Fredholm, then the differential equation  $\dot{x} = A(t)x$  has an exponential dichotomy on both  $[0, \infty)$  and  $(-\infty, 0]$ . This gives a converse to an earlier result.

**1. Introduction.** In [4] it was shown that if the linear system

$$(1) \quad \dot{x} = A(t)x$$

has an exponential dichotomy on  $[0, \infty)$  and  $(-\infty, 0]$ , then the operator

$$L: BC^1(-\infty, \infty) \rightarrow BC(-\infty, \infty)$$

(see below for notations) defined by

$$(2) \quad (Lx)(t) = \dot{x}(t) - A(t)x(t)$$

is Fredholm. The object of this note is to prove a converse to this.

**2. Statement of the theorem.** We denote by  $A(t)$  an  $n \times n$  matrix-valued function, bounded and continuous on an interval  $J$ , and by  $X(t)$  a fundamental matrix for system (1). System (1) is said to have an *exponential dichotomy* on  $J$  if there is a projection  $P(P^2 = P)$  and constants  $K \geq 1$ ,  $\alpha > 0$  such that

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq Ke^{-\alpha(t-s)} & (s \leq t), \\ |X(t)(I - P)X^{-1}(s)| &\leq Ke^{-\alpha(s-t)} & (s \geq t). \end{aligned}$$

We denote by  $BC(J)$  the Banach space of bounded continuous functions  $x: J \rightarrow \mathbf{R}^n$  with the norm

$$\|x\| = \sup_{t \in J} |x(t)|$$

and by  $BC^1(J)$  the Banach space of continuously differentiable functions  $x: J \rightarrow \mathbf{R}^n$  bounded together with their derivatives, where in the latter space we use the norm

$$\|x\|_1 = \sup_{t \in J} |x(t)| + \sup_{t \in J} |\dot{x}(t)|.$$

With system (1) we associate the bounded linear operator  $L: BC^1(J) \rightarrow BC(J)$  defined by (2) and denote by  $\mathcal{N}(L)$  and  $R(L)$  its nullspace and range. In Lemma

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4.2 in Palmer [4] it was shown that if equation (1) has exponential dichotomies on both  $[0, \infty)$  and  $(-\infty, 0]$ , then the operator  $L$  is Fredholm. Moreover it was shown that a function  $f \in BC((-\infty, \infty))$  is in  $R(L)$  if and only if  $\int_{-\infty}^{\infty} \psi^*(t)f(t) dt = 0$  (\* denotes transpose) for all bounded solutions  $\psi(t)$  of the adjoint equation

$$(3) \quad \dot{x} = -A^*(t)x;$$

also the index of  $L$  is the difference between the ranks of the projections associated with the dichotomies on  $[0, \infty)$  and  $(-\infty, 0]$ .

**THEOREM.** *Let  $A(t)$  be an  $n \times n$  matrix-valued function, bounded and continuous on an interval  $J$ , where  $J = (-\infty, \infty)$ ,  $[0, \infty)$  or  $(-\infty, 0]$ . Suppose the operator  $L: BC^1(J) \rightarrow BC(J)$  defined by (2) is semi-Fredholm. Then when  $J$  is a half-line, system (1) has an exponential dichotomy on  $J$  and when  $J = (-\infty, \infty)$ , (1) has exponential dichotomies on both  $[0, \infty)$  and  $(-\infty, 0]$ .*

**REMARK 1.** The nullspace  $\mathcal{N}(L)$  of  $L$  is always finite-dimensional and complemented. For let  $P$  be a projection with range  $\{\xi \in \mathbf{R}^n: \sup_{t \in J} |X(t)X^{-1}(0)\xi| < \infty\}$ . Then  $\mathcal{N}(L)$  is the set of functions  $X(t)\xi$  with  $\xi$  in the range of  $P$  and hence  $\mathcal{N}(L)$  is finite-dimensional. Also  $Y = \{x \in BC^1(J): Px(0) = 0\}$  is a complement to  $\mathcal{N}(L)$ . So the assumption that  $L$  is semi-Fredholm (see Kato [2, p. 230]) means simply that the range  $R(L)$  of  $L$  is closed.

**REMARK 2.** When  $L$  is surjective the result is already known (cf. Massera and Schäffer [3], Coppel [1]).

**REMARK 3.** The theorem shows that when  $L$  is semi-Fredholm it must be, in fact, Fredholm and even surjective in the half-line case.

Thanks are due to Andrew Coppel for his suggestions concerning the work in this paper.

**3. Proof of the Theorem.** To prove the theorem we need two lemmas.

**LEMMA 1.** *Let  $A(t)$  be an  $n \times n$  matrix-valued function, bounded and continuous on an interval  $J$ , when  $J = (-\infty, \infty)$ ,  $[0, \infty)$  or  $(-\infty, 0]$ . Let  $f \in BC(J)$  be of compact support. Then if  $J$  is a half-line, the equation*

$$(4) \quad \dot{x} = A(t)x + f(t)$$

*has a solution  $x(t)$  of compact support. If  $J = (-\infty, \infty)$  equation (4) has a solution  $x(t)$  of compact support if and only if  $\int_{-\infty}^{\infty} \psi^*(t)f(t) dt = 0$  for all solutions  $\psi(t)$  of the adjoint equation (3).*

**PROOF.** First let  $J = [0, \infty)$ . Then any solution  $x(t)$  of equation (4) can be written as

$$(5) \quad x(t) = X(t)X^{-1}(0)\{\xi + X(0) \int_0^t X^{-1}(s)f(s) ds\}.$$

Since  $f$  has compact support, there exists  $\tau \geq 0$  such that  $f(t) = 0$  for  $t \geq \tau$ . Then we see that for  $t \geq \tau$ ,

$$x(t) = X(t)X^{-1}(0)\{\xi + X(0) \int_0^{\infty} X^{-1}(s)f(s) ds\}.$$

So  $x(t)$  has compact support if and only if  $\xi = -X(0) \int_0^{\infty} X^{-1}(s)f(s) ds$ . This proves the lemma for  $J = [0, \infty)$ . The proof for  $J = (-\infty, 0]$  is similar.

Now let  $J = (-\infty, \infty)$ . Then any solution  $x(t)$  of equation (4) has the form (5).  $x(t)$  has compact support in  $[0, \infty)$  if and only if  $\xi = -X(0) \int_0^\infty X^{-1}(s)f(s) ds$  and compact support in  $(-\infty, 0]$  if and only if  $\xi = X(0) \int_{-\infty}^0 X^{-1}(s)f(s) ds$ . These two values are the same if and only if  $\int_{-\infty}^\infty X^{-1}(s)f(s) ds = 0$ . Since the transposes of the rows of  $X^{-1}(t)$  form a basis for the solution space of the adjoint equation (3), this last condition is equivalent to the condition that  $\int_{-\infty}^\infty \psi^*(s)f(s) ds = 0$  for all solutions  $\psi(t)$  of the adjoint equation (3). This completes the proof of the lemma.

Now we take

$$\mathcal{F} = C_0(J) = \{f \in BC(J): f(t) \rightarrow 0 \text{ as } |t| \rightarrow \infty\}.$$

This is a closed subspace of  $BC(J)$  in which the functions of compact support are dense. Since  $L$  is continuous,  $\xi = L^{-1}(\mathcal{F})$  is also a closed subspace of  $BC^1(J)$ . Then we define  $T: \mathcal{E} \rightarrow \mathcal{F}$  to be the restriction of  $L$  to  $\mathcal{E}$ . In the following lemma, we characterize  $\mathcal{N}(T^*)$ , where  $T^*: \mathcal{F}^* \rightarrow \mathcal{E}^*$  is the conjugate operator.

LEMMA 2. Let  $A(t), J$  be as in Lemma 1, and let  $T: \mathcal{E} \rightarrow \mathcal{F}$  be as just defined. Then if  $J$  is a half-line,  $\mathcal{N}(T^*) = \{0\}$ . If  $J = (-\infty, \infty)$ ,  $\alpha \in \mathcal{N}(T^*)$  if and only if there is a solution  $\psi(t)$  of the adjoint equation (3) with  $\int_{-\infty}^\infty |\psi(t)| dt < \infty$  such that for all  $f$  in  $\mathcal{F}$

$$(6) \quad \alpha(f) = \int_{-\infty}^\infty \psi^*(t)f(t) dt.$$

In particular, this means that  $\dim \mathcal{N}(T^*) < \infty$ .

PROOF. Let  $J$  be a half-line and let  $f \in BC(J)$  be of compact support. Then by Lemma 1  $f$  is in  $R(L)$  and hence in  $R(T)$ . So  $f = Tx$  for some  $x \in \mathcal{E}$ . Therefore, if  $\alpha \in \mathcal{N}(T^*)$ ,

$$\alpha(f) = \alpha(Tx) = (T^*\alpha)(x) = 0.$$

This holds for all functions  $f$  in  $\mathcal{F}$  of compact support; these functions are dense in  $\mathcal{F}$  and  $\alpha$  is continuous. So  $\alpha = 0$ , as required.

Now let  $J = (-\infty, \infty)$  and suppose  $\alpha \in \mathcal{N}(T^*)$ . The following argument was suggested by Shilov [5, p. 25]. Let  $\phi(t)$  be a positive function of compact support with  $\int_{-\infty}^\infty \phi(t) dt = 1$ . Let  $f \in \mathcal{F}$  be of compact support and define

$$(7) \quad \tilde{f}(t) = f(t) - \phi(t)X(t) \int_{-\infty}^\infty X^{-1}(s)f(s) ds.$$

Then we see that  $\tilde{f}(t)$  has compact support and  $\int_{-\infty}^\infty X^{-1}(t)\tilde{f}(t) dt = 0$ . So by Lemma 1,  $\tilde{f}$  is in  $R(L)$  and hence in  $R(T)$ . That is,  $\tilde{f} = Tx$  for some  $x$  in  $\mathcal{E}$ . Then since  $\alpha \in \mathcal{N}(T^*)$ ,

$$(8) \quad \alpha(\tilde{f}) = \alpha(Tx) = (T^*\alpha)(x) = 0.$$

Then it follows from (7), (8) and a simple calculation that  $\alpha(f) = \int_{-\infty}^\infty \psi^*(t)f(t) dt$ , where

$$\psi(t) = \sum_{i=1}^n \alpha(\phi(\cdot)x_i(\cdot))\psi_i(t),$$

$x_i(t)$  being the  $i$ th column of  $X(t)$  and  $\psi_i(t)$  the transpose of the  $i$ th row of  $X^{-1}(t)$ .

Certainly  $\psi(t)$  is a solution of the adjoint equation (3) and, using (6), for all functions  $f \in \mathcal{F}$  of compact support,

$$\left| \int_{-\infty}^{\infty} \psi^*(t)f(t) dt \right| = |\alpha(f)| \leq \|\alpha\| \|f\|.$$

It follows that  $\int_{-\infty}^{\infty} |\psi(t)| dt \leq \|\alpha\| < \infty$ . Then  $\alpha$  and  $f \rightarrow \int_{-\infty}^{\infty} \psi^*(t)f(t) dt$  are both bounded linear functionals defined on  $\mathcal{F}$  and coinciding on the dense subset consisting of the functions of compact support. So (6) holds for all  $f \in \mathcal{F}$ , as required.

Conversely, suppose  $\psi(t)$  is a solution of the adjoint equation (3) satisfying  $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$ . Then

$$\int_{-\infty}^{\infty} |\dot{\psi}(t)| dt = \int_{-\infty}^{\infty} |A^*(t)\psi(t)| dt \leq \sup_{-\infty < t < \infty} |A^*(t)| \int_{-\infty}^{\infty} |\psi(t)| dt < \infty$$

so that  $\psi(t) = \psi(0) + \int_0^t \dot{\psi}(s) ds$  has limits as  $t \rightarrow \pm\infty$ . Since  $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$ , these limits must both be zero. So  $\psi(t) \rightarrow 0$  as  $|t| \rightarrow \infty$ .

Now  $\alpha$  defined by (6) is certainly in  $\mathcal{F}^*$ . Moreover, if  $x \in \mathcal{E}$ ,

$$\begin{aligned} (T^*\alpha)(x) &= \alpha(Tx) = \int_{-\infty}^{\infty} \psi^*(t)[\dot{x}(t) - A(t)x(t)] dt \\ &= [\psi^*(t)x(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [\dot{\psi}^*(t) + \psi^*(t)A(t)]x(t) dt \\ &\hspace{15em} \text{(integrating by parts)} \\ &= 0. \end{aligned}$$

**PROOF OF THE THEOREM.** We are supposing that  $L$  is semi-Fredholm. So  $R(L)$  is closed. Hence  $R(T)$  is closed also. Then by Theorem 4.6—C in Taylor [6, p. 226],

$$\{R(T)\}^0 := \{\alpha \in \mathcal{F}^* : \alpha(f) = 0 \forall f \in R(T)\} = \mathcal{N}(T^*).$$

Suppose now  $J$  is a half-line. Then by Lemma 2,  $\mathcal{N}(T^*) = \{0\}$ . So by the Hahn-Banach theorem,  $R(T) = \mathcal{F}$ . That is, for all  $f \in BC(J)$  with  $f(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$  (+ when  $J = [0, \infty)$ , - when  $J = (-\infty, 0]$ ), equation (4) has a solution bounded on  $J$ . Then it follows from Theorem 64.B in Massera and Schäffer [3, p. 189] or from the proof of Proposition 3 in Coppel [1, p. 22] that equation (1) has an exponential dichotomy on  $J$ .

Suppose now  $J = (-\infty, \infty)$ . It is an exercise to prove that since  $\{R(T)\}^0 = \mathcal{N}(T^*)$  and  $\dim \mathcal{N}(T^*) < \infty$ , then  $R(T)$  is saturated (Taylor [6, p. 225]). That is,

$$R(T) = {}^0\{ \mathcal{N}(T^*) \} := \{f \in \mathcal{F} : \alpha(f) = 0 \forall \alpha \in \mathcal{N}(T^*)\}.$$

By Lemma 2 this means that  $f \in R(T)$  if and only if  $\int_{-\infty}^{\infty} \psi^*(t)f(t) dt = 0$  for all solutions  $\psi(t)$  of the adjoint equation (3) satisfying  $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$ .

Now let  $\psi_1(t), \psi_2(t), \dots, \psi_m(t)$  be a basis for the subspace of solutions  $\psi(t)$  of equation (3) satisfying  $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$ . Let  $f \in C_0([0, \infty))$ . We seek  $g \in C_0((-\infty, 0])$  such that

$$(9) \quad \int_{-\infty}^0 \psi_i^*(t)g(t) dt = - \int_0^{\infty} \psi_i^*(t)f(t) dt \quad (i = 1, \dots, m),$$

$$(10) \quad g(0) = f(0).$$

Such a  $g$  certainly exists provided the bounded linear functionals on  $C_0((-\infty, 0])$  defined by

$$\begin{aligned} \alpha_i(g) &= \int_{-\infty}^0 \psi_i^*(t)g(t) dt \quad (i = 1, \dots, m), \\ \beta_j(g) &= g_j(0) \quad (j = 1, \dots, n), \end{aligned}$$

where  $g_j(t)$  is the  $j$ th component of  $g(t)$ , are linearly independent.

Suppose there are scalars  $\gamma_i$  ( $i = 1, \dots, m$ ),  $\delta_j$  ( $j = 1, \dots, n$ ) such that  $\sum_{k=1}^m \gamma_k \alpha_k = \sum_{j=1}^n \delta_j \beta_j$ . Then for all  $g$  in  $C_0((-\infty, 0])$ ,

$$(11) \quad \int_{-\infty}^{\infty} \psi^*(t)g(t) dt = \sum_{j=1}^n \delta_j g_j(0),$$

where  $\psi(t) = \sum_{i=1}^m \gamma_i \psi_i(t)$ . Suppose  $\psi \neq 0$ . Then we can choose  $g$  so that  $\psi^*(t)g(t) > 0$  for all  $t$  but  $g(0) = 0$ . Then the left side of (11) would be positive but the right side zero. So  $\psi = 0$ . Since the  $\psi_i$ 's are linearly independent, it follows that  $\gamma_i = 0$  for all  $i$ . Then for all  $g$ ,  $\sum_{j=1}^n \delta_j g_j(0) = 0$ . For each  $j$  we can choose  $g$  so that  $g_j(0) = 1$  but  $g_k(0) = 0$  for  $k \neq j$ . So we conclude that  $\delta_j = 0$  for all  $j$  also. Hence the linear functionals are linearly independent and so  $g \in C_0((-\infty, 0])$  can be chosen so that (9) and (10) are satisfied.

Now we define  $\tilde{f}(t)$  to be  $f(t)$  for  $t \geq 0$  and  $g(t)$  for  $t < 0$ . Then  $\tilde{f} \in C_0((-\infty, \infty))$  and  $\int_{-\infty}^{\infty} \psi_i^*(t)f(t) dt = 0$  for  $i = 1, \dots, m$ . So  $\tilde{f} \in R(T)$ . This means that the equation  $\dot{x} = A(t)x + \tilde{f}(t)$  has a solution bounded on  $(-\infty, \infty)$ . Restricting to  $[0, \infty)$  we conclude that equation (4) has a solution bounded on  $[0, \infty)$ . This holds for all  $f$  in  $C_0([0, \infty))$  and so, by the result of Massera and Schäffer used earlier, it follows that equation (1) has an exponential dichotomy on  $[0, \infty)$ . A similar argument shows that it has an exponential dichotomy on  $(-\infty, 0]$ . So the proof of the theorem is complete.

**4. The effect of perturbations.** If  $\tilde{L}$  is a small perturbation of a Fredholm operator  $L$ , then  $\tilde{L}$  is also Fredholm, has the same index as  $L$  and  $\dim \mathcal{N}(\tilde{L}) \leq \dim \mathcal{N}(L)$ . We want to show that both the possibilities  $\dim \mathcal{N}(\tilde{L}) = \dim \mathcal{N}(L)$  and  $\dim \mathcal{N}(\tilde{L}) < \dim \mathcal{N}(L)$  do arise in the case of differential equations.

First we look at an abstract situation.

**PROPOSITION.** *Let  $\mathcal{E}, \mathcal{F}$  be Banach spaces,  $L: \mathcal{E} \rightarrow \mathcal{F}$  a Fredholm operator of index zero with  $\dim \mathcal{N}(L) = 1$  and let  $N_i: \mathcal{E} \rightarrow \mathcal{F}$  ( $i = 1, 2$ ) be bounded linear operators.*

*Suppose that not both  $N_1\phi$  and  $N_2\phi$  are in  $R(L)$ , where  $\phi \in \mathcal{N}(L)$ ,  $\phi \neq 0$ . Then there is a neighborhood  $O$  of the origin in  $\mathbf{R}^2$  and a curve passing through the origin such that when  $(\mu_1, \mu_2) \in O$ ,  $\dim \mathcal{N}(L + \mu_1 N_1 + \mu_2 N_2) = 1$  if  $(\mu_2, \mu_2)$  is on the curve and  $= 0$  otherwise.*

**PROOF.** The proof is a straightforward application of the Lyapunov-Schmidt method.

Let  $\alpha$  be a bounded linear functional on  $\mathcal{E}$  such that  $\alpha(\phi) = 1$ . Also let  $\beta$  be a bounded linear functional on  $\mathcal{F}$  such that  $R(L) = \mathcal{N}(\beta)$  and let  $\eta \in F$  satisfy  $\beta(\eta) = 1$ . We seek those values of  $(\mu_1, \mu_2)$  for which the equation

$$(L + \mu_1 N_1 + \mu_2 N_2)x = 0$$

has nonzero solutions  $x \in \mathcal{E}$ . We may write  $x = \gamma\phi + w$  where  $\gamma$  is real and  $w \in \mathcal{N}(\alpha)$ . Then we want to solve

$$Lw = -(\mu_1 N_1 + \mu_2 N_2)(\gamma\phi + w)$$

for  $\gamma$  and  $w$ . Solution of this equation is equivalent to solution of the pair of equations

$$(12) \quad Lw = -Q(\mu_1 N_1 + \mu_2 N_2)(\gamma\phi + w),$$

$$(13) \quad \beta((\mu_1 N_1 + \mu_2 N_2)(\gamma\phi + w)) = 0,$$

where  $Q$  is the projection on  $\mathcal{F}$  defined by  $Q(y) = y - \beta(y)\eta$ . Note that  $R(Q) = R(L)$ .

By the bounded inverse theorem,  $L: \mathcal{N}(\alpha) \rightarrow R(L)$  is invertible. So when  $\mu_1, \mu_2$  are small, the operator  $K(\mu_1, \mu_2) = L + Q(\mu_1 N_1 + \mu_2 N_2): \mathcal{N}(\alpha) \rightarrow R(L)$  is also invertible. So we can solve (12) for  $w = \gamma w(\mu_1, \mu_2)$ , where  $w(\mu_1, \mu_2) = -K(\mu_1, \mu_2)^{-1}Q(\mu_1 N_1 + \mu_2 N_2)(\phi)$ . Note that  $w(\mu_1, \mu_2)$  is smooth with  $w(0, 0) = 0$ . Then we substitute  $w = \gamma w(\mu_1, \mu_2)$  into equation (13) to get an equation

$$\gamma h(\mu_1, \mu_2) = 0,$$

where

$$h(\mu_1, \mu_2) = \beta((\mu_1 N_1 + \mu_2 N_2)(\phi + w(\mu_1, \mu_2))).$$

Now  $x = \gamma(\phi + w(\mu_1, \mu_2))$  where  $w(0, 0) = 0$ . So if  $\mu_1, \mu_2$  are sufficiently small,  $x \neq 0$  if and only if  $\gamma \neq 0$ . Thus  $\mathcal{N}(L + \mu_1 N_1 + \mu_2 N_2) \neq \{0\}$  if and only if  $h(\mu_1, \mu_2) = 0$ .

Furthermore,  $h(0, 0) = 0$  and  $\partial h(0, 0)/\partial \mu_i = \beta(N_i \phi)$  ( $i = 1, 2$ ). Since  $N_i \phi \notin R(L)$  for some  $i$ ,  $\partial h(0, 0)/\partial \mu_i \neq 0$  for some  $i$ . So by the implicit function theorem there is a neighborhood  $O$  of the origin in  $\mathbf{R}^2$  and a curve through the origin such that when  $(\mu_1, \mu_2) \in O$ ,  $h(\mu_1, \mu_2) = 0$  if and only if  $(\mu_1, \mu_2)$  lies on the curve. This completes the proof of the proposition.

EXAMPLE. Let  $g(x)$  be a  $C^1$  real function with  $g(0) = 0$ ,  $g'(0) < 0$ . This implies that the origin (in phase space) is a saddle point for the equation

$$\ddot{x} + g(x) = 0.$$

We also suppose the equation has a homoclinic solution; that is, a solution  $\zeta(t) \rightarrow 0$  as  $|t| \rightarrow \infty$ . Then stable manifold theory tells us that  $\zeta(t)$  and  $\dot{\zeta}(t) \rightarrow 0$  exponentially as  $|t| \rightarrow \infty$ .

We consider the variational equation

$$\ddot{x} + g'(\zeta(t))x = 0$$

and its perturbation

$$\ddot{x} + \mu_1 \dot{x} + [g'(\zeta(t)) + \mu_2]x = 0.$$

As a system this has the form

$$\dot{x} = [A(t) + \mu_1 B_1 + \mu_2 B_2]x,$$

where  $x \in \mathbf{R}^2$  and

$$A(t) = \begin{bmatrix} 0 & 1 \\ -g'(\zeta(t)) & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

The corresponding operator from  $BC^1((-\infty, \infty))$  into  $BC((-\infty, \infty))$  has the form  $L + \mu_1 N_1 + \mu_2 N_2$ , where  $L$  is defined by (2) and  $(N_i x)(t) = B_i x(t)$  ( $i = 1, 2$ ).

Now system (1) with  $A(t)$  as above has the property that  $A(t) \rightarrow C = \begin{bmatrix} 0 & 1 \\ -g'(0) & 0 \end{bmatrix}$  as  $|t| \rightarrow \infty$ . The eigenvalues of  $C$  are  $\pm \sqrt{-g'(0)}$  and so  $\dot{x} = Cx$  has an exponential dichotomy on  $(-\infty, \infty)$  with projection of rank 1. It follows from the roughness theorem (Coppel [1, p. 34]) that system (1) has exponential dichotomies on intervals  $[\tau, \infty)$  and  $(-\infty, -\tau]$  for  $\tau$  sufficiently large and hence on both half-lines by Coppel [1, p. 13]. Also the projections involved have rank 1. Then it follows from Lemma 4.2 in Palmer [4] that the operator  $L: BC^1((-\infty, \infty)) \rightarrow BC((-\infty, \infty))$  defined by (2) is Fredholm of index zero. Next we see by inspection that

$$\phi(t) = \begin{bmatrix} \dot{\zeta}(t) \\ \ddot{\zeta}(t) \end{bmatrix}$$

is a solution of equation (1) bounded on  $(-\infty, \infty)$ . Since the dimensions of the subspaces of solutions bounded on the half-lines are 1, it follows that  $\phi(t)$  is, up to a scalar multiple, the unique solution of (1) bounded on  $(-\infty, \infty)$ . Hence  $\dim \mathcal{N}(L) = 1$ .

We also see by inspection that

$$\psi(t) = \begin{bmatrix} \ddot{\zeta}(t) \\ -\dot{\zeta}(t) \end{bmatrix}$$

is a bounded solution of the adjoint equation (3). From the proof of Lemma 4.2 in Palmer [4], the subspaces of the bounded solutions (on  $(-\infty, \infty)$ ) of equations (1) and (3) have the same dimension (in this case). So up to a scalar multiple,  $\psi(t)$  is the unique bounded solution of (3). Moreover it follows from the same lemma that

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

is in the range of  $L$  if and only if

$$\int_{-\infty}^{\infty} \psi^*(t)f(t) dt = \int_{-\infty}^{\infty} [\ddot{\zeta}(t)f_1(t) - \dot{\zeta}(t)f_2(t)] dt = 0.$$

Now

$$\begin{aligned} \int_{-\infty}^{\infty} \psi^*(t)(N_1 \phi)(t) dt &= \int_{-\infty}^{\infty} \ddot{\zeta}(t)\dot{\zeta}(t) dt = 0, \\ \int_{-\infty}^{\infty} \psi^*(t)(N_2 \phi)(t) dt &= \int_{-\infty}^{\infty} \dot{\zeta}(t)^2 dt > 0. \end{aligned}$$

So  $N_2 \phi \notin R(L)$  and the Proposition implies there is a neighborhood  $O$  of  $(0, 0)$  in  $\mathbf{R}^2$  and a curve passing through the origin such that when  $(\mu_1, \mu_2) \in O$ ,  $\dim \mathcal{N}(L + \mu_1 N_1 + \mu_2 N_2) = 1$  if  $(\mu_1, \mu_2)$  is on the curve and  $= 0$  otherwise.

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