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LANDAU'S INEQUALITY FOR THE DIFFERENCE OPERATOR

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ABSTRACT. The best constants for Landau's inequality with the classical p-norms are known explicitly only when $p = 1, 2$ and ∞. This is true for both the discrete and the continuous versions of the inequality and for both the "whole line" and "half line" cases. In each of the six known cases the best constant in the discrete version is the same as the best constant for the continuous version. Here we show that for many other values of p the discrete constants are strictly greater than the corresponding continuous ones. In addition, we show that the "three norm version" of the inequality, established by Nirenberg and Gabushin in the continuous case is also valid in the discrete case.

1. Introduction. The inequality under investigation here is

$$||\Delta^k x||_q \leq C||x||_p^\alpha ||\Delta^n x||_r^\beta. \tag{1.1}$$

Throughout this paper we assume that k and n are integers satisfying $1 \leq k < n$, $1 \leq q, p, r \leq \infty$. For $x = \{x_j\}_{j\in M}$ where $M = Z = \{\cdots -2, -1, 0, 1, 2, \dots\}$ or $M = Z^+ = \{0, 1, 2, \dots\}$ we have

$$||x||_p^p = \sum_{j\in M} |x_j|^p, \qquad 1 \leq p < \infty,$$

$$||x||_\infty = \sup |x_j|, \quad j \in M, \qquad p = \infty.$$

The operator Δ is the classical difference operator

$$\Delta x = \{x_{j+1} - x_j\}_{j\in M}, \qquad \Delta^n = \Delta(\Delta^{n-1}), \qquad n = 2, 3, \dots; \tag{1.2}$$
$$\alpha = (n - k - r^{-1} + q^{-1})/(n - r^{-1} + p^{-1}), \qquad \beta = 1 - \alpha,$$

with the usual convention that $s^{-1} = 0$ when $s = \infty$.

Inequality (1.1) is the discrete analogue of the well-known Nirenberg [**1955**] and Gabushin [**1967**] continuous version

$$||y^{(k)}||_q \leq K||y||_p^\alpha ||y^{(n)}||_r^\beta. \tag{1.3}$$

In (1.3) the norm is the classical $L^p(J)$ norm with $J = R = (-\infty, \infty)$ or $J = R^+ = (0, \infty)$:

$$||y||_p^p = \int_J |y(t)|^p \, dt, \qquad 1 \leq p < \infty,$$

$$||y||_\infty = \operatorname{ess\,sup} |y(t)|, \qquad t \in J.$$

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Nirenberg [**1955**] and Gabushin [**1967**] have shown that for given p, q, r, $1 \leq p, q, r \leq \infty$ and given $J = R$ or R^+ and α, β given by (1.2) there exists a positive number K such that (1.3) holds for all y in $W^n(p, r; J) = \{y \in L^p(J) \colon y^{(n-1)}$ is locally absolutely continuous on J and $y^{(n)} \in L^r(J)\}$ if and only if

$$nq^{-1} \leq (n-k)p^{-1} + kr^{-1}. \tag{1.4}$$

Clearly if (1.3) holds with some positive constant K there is a smallest such constant which we denote by $K = K(n, k, q, p, r; J)$ to emphasize its dependence on these quantities. The names of Landau, Hardy-Littlewood and Kolmogorov are often attached to the following special cases of (1.3): (i) $n = 2$, $k = 1$, $p = q = r = \infty$, $J = R^+$, (ii) $n = 2$, $k = 1$, $p = q = r = 2$, $J = R^+$ and (iii) general n, k, $p = q = r = \infty$, $J = R$, respectively.

Similarly the best constant in (1.1) is denoted by $C = C(n, k, q, p, r; M)$ with $M = Z$ or Z^+.

The major purpose of this paper is twofold:

1. To show that $C(2, 1, p, p, p; Z) > K(2, 1, p, p, p; R)$ for many values of p in the range $3 < p < \infty$.

2. There exists a finite constant C such that (1.1) is valid if and only if there exists a finite K in (1.3).

REMARK. We say that inequality (1.1) holds or is valid if (and only if) for given integers n, k, $1 \leq k < n$, given q, p, r, $1 \leq p, q, r \leq \infty$ and given $M = Z$ or Z^+ there exists a positive number C such that (1.1) holds for all x in $l^p(M)$ satisfying $\Delta^n x \in l^r(M)$.

We also raise a number of questions about these inequalities.

2. The second-order case.

In this section we assume that $n = 2$, $k = 1$ and $q = p = r$ and use the notation $C(p, M) = C(2, 1, p, p, p; M)$, $K(p, J) = K(2, 1, p, p, p; J)$. Even in these special cases the best constants are known explicitly only for $p = 1, 2, \infty$. These results are summarized in

THEOREM 1. $C(2, Z) = 1 = K(2, R)$, $C(1, Z^+) = \sqrt{5/2} = K(1, R^+)$, $C(\infty, Z^+) = 2$, $C(1, Z) = C(\infty, Z) = C(2, Z^+) = \sqrt{2} = K(1, R) = K(\infty, R) = K(2, R^+)$.

PROOF. Proofs of these results can be found in the references given in Kwong and Zettl [**1980**].

It is interesting to note that the discrete constants are the same as the corresponding continuous constants in all six cases where they are known explicitly. Thus one might be tempted to conjecture that $C(p, Z) = K(p, R)$ and $C(p, Z^+) = K(p, R^+)$ for all p, $1 \leq p \leq \infty$. That this is not so, at least for the whole line case, is shown by

THEOREM 2. *For some values of* $p > 3$ *we have* $C(p, Z) > K(p, R)$.

PROOF. First we establish

LEMMA 1. *For* $1 \leq p \leq 2$ *we have*

$$C(p, Z) \geq 2^{2/p - 1} \tag{2.1}$$

and for $2 \leq p < \infty$ *we have*

$$C(p, Z) \geq 2^{1 - 2/p}. \tag{2.2}$$

PROOF OF LEMMA 1. A sequence $x = (x_j)$ is said to be P-periodic if P is a positive integer such that for all j in Z

$$x_{j+P} = x_j.$$

For such a sequence x we define its "periodic l^p norm" as

$$||x||_{p,P} = \left(\sum_{j=0}^{P-1} |x_j|^p \right)^{1/p}, \qquad 1 \le p < \infty.$$

Note that if x is P-periodic then so is Δx.

By Theorem 8 of Kwong and Zettl [**1988**] we have

$$C(p, Z) = \sup \frac{||\Delta x||^2_{p,P}}{||x||_{p,P} ||\Delta^2 x||_{p,P}} \tag{2.3}$$

where the supremum is taken over all nonzero P-periodic sequences x in $l^\infty(Z)$ with $\Delta^2 x \neq 0$ for all $P = 1, 2, 3, \ldots$.

Applying (2.3) to the 4-periodic sequences $\ldots 0\ 1\ 0\ -1 \ldots$ and $\ldots 1\ 1\ -1\ -1 \ldots$ we get (2.1) and (2.2), respectively. (Actually both (2.1) and (2.2) hold for all p, $1 \le p < \infty$ but are interesting only for the ranges of p indicated.)

It was shown in Franco, Kaper, Kwong and Zettl [**1983**] that

$$K(p, R) \le U(p)$$

where

$$U(p) = (q-1)^{(2-q^n)q^{-n}} \left(\prod_{i=1}^{n} \left(\frac{q}{q^i - 1} - 1 \right)^{q^{-i}} \right)^{2(q-1)}, \tag{2.4}$$

$2 < p < \infty$, $p^{-1} + q^{-1} = 1$, $n = [(\log_2 q)^{-1}]$ and $[\cdot]$ is the greatest integer function.

From (2.2) and (2.3) (see also Franco, Kaper, Kwong and Zettl [**1983**, p. 261]), for $p = 4$ we get

$$K(4, R) \le U(4) = (15/7)^{3/8} \doteq 1.33082962 < 2^{1/2} \doteq 1.414214 \le C(4, Z).$$

Similarly, we obtain

$$\begin{aligned} K(5, R) \le U(5) &= 4^{47/125} (11/9)^{8/25} (19/61)^{32/125} \\ &\doteq 1.33222966 < 2^{1-2/5} \doteq 1.515717 \le C(5, Z). \end{aligned}$$

$$\begin{aligned} K(6, R) \le U(6) &= 5^{19/108} (19/11)^{5/18} (59/91)^{25/108} \\ &\doteq 1.39745611 < 2^{1-2/6} \doteq 1.587401 \le C(6, Z). \end{aligned}$$

REMARK 1. Numerical evidence strongly suggests that $C(p, Z) > K(p, R)$ for all p satisfying $3 < p < \infty$. In fact this follows from the lower bound (2.2) for $C(p, Z)$ and the upper bound $U(p)$ for $K(p, R)$ for every p in the range $3 < p < \infty$ for which we have made the computation including values of p up to $p = 10^5$. A natural question to ask is:

QUESTION 1. Is $C(p, Z) > K(p, R)$ for $1 < p < 2$ and for $2 < p < \infty$?

It seems to us that the upper bound $U(p)$ of $K(p, R)$ is "good" when $p > 3$ but not so good when $p < 3$. We expect that before question 1 is answered for $1 < p < 2$ and $2 < p < 3$ a better upper bound than $U(p)$ needs to be found.

The "half line" version of question 1 is

QUESTION 2. Is $C(p, Z^+) > K(p, R^+)$ for $1 < p < 2$ and $2 < p < \infty$?

In answering question 2 one is hampered by the lack of "good" upper bounds for $K(p, R^+)$ and "good" lower bounds for $C(p, Z^+)$.

There seem to be no upper bounds comparable to $U(p)$ known for $K(p, R^+)$ and no lower bounds comparable to (2.1), (2.2) known for $C(p, Z^+)$.

Can the lower bounds (2.1), (2.2) be improved? or

QUESTION 3. Is $C(p, Z) = 2^{1-2/p}$ for $2 < p < \infty$?

QUESTION 4. Is $C(p, Z) = 2^{2/p-1}$ for $1 < p < 2$?

3. The discrete inequality with three norms. Here we establish the discrete analogue (1.1) of the well known inequality (1.3). The proof of the continuous result (1.3) does not seem to be extendable to the discrete case. Here we use the continuous result to prove the discrete case.

THEOREM 3. *Let k and n be integers with $1 \leq k < n$; let p, q, r satisfy $1 \leq p, q, r \leq \infty$; let α, β be given by* (1.2); *let $M = Z$ or $M = Z^+$. Then there exists a positive number C such that*

$$||\Delta^k x||_q \leq C ||x||_p^\alpha ||\Delta^n x||_r^\beta \tag{1.1}$$

for all x in $l^p(M)$ satisfying $\Delta^n x \in l^r(M)$ if and only if

$$nq^{-1} \leq (n-k)p^{-1} + kr^{-1}. \tag{1.4}$$

PROOF. Using standard approximation arguments it can be shown that

$$C\left(n, k, q, p, r; \begin{matrix} Z \\ Z^+ \end{matrix}\right) \geq K\left(n, k, q, p, r; \begin{matrix} R \\ R^+ \end{matrix}\right). \tag{3.1}$$

This is done in Ditzian [**1983**], see also Kaper-Spellman [**1987**], for the case $q = p = r = \infty$ with Z and R. Since the proof for the general case is similar, we omit the details. Thus if (1.4) fails then $C \geq K = \infty$. By this we mean that (1.3) and consequently (1.1) are not valid. Assume condition (1.4) is satisfied. We will show that (1.3) implies (1.1). For this it is sufficient to prove the case $n = 2$ since $n > 2$ then follows by induction. (The induction argument is not completely straightforward—see Kwong and Zettl [**1980**] for details.) We proceed with the "whole line" version of $n = 2$ i.e. $M = Z$. The case $M = Z^+$ is similar and hence omitted.

To relate the discrete case to the continuous case we use a construction due to Ditzian [**1983**].

Given a sequence $x = \{x_j\}_{j \in M}$ define a function $f = Tx$ on R by

$$f(t) = \sum_{j \in M} x_j B_{j,3}(t), \qquad t \in R, \tag{3.2}$$

where $B_{j,3}$ is the B-spline of order 3 with support in $[j, j+3]$. See de Boor [**1978**, Chapter IX] for a discussion of B-splines. Then f' is the piecewise linear interpolant of Δx and f'' is the piecewise constant interpolant of $\Delta^2 x$ with constant value in each $(j, j+1)$.

$$f'(t) = \sum_{j \in M} (\Delta x)_{j-1} B_{j,2}(t),$$

$$f''(t) = \sum_{j \in M} (\Delta^2 x)_{j-2} B_{j,1}(t).$$

Thus if $\Delta^2 x \in l^r(Z)$ then $f'' \in L^r(R)$ and

$$||f''||_r = ||\Delta^2 x||_r. \tag{3.3}$$

Now we show that there is a positive number A, independent of x and f, such that

$$||f||_p \leq A||x||_p. \tag{3.4}$$

Note that each of $B_{j,3}(t)$ is a translate of $B_{0,3}$ and has support in an interval of length 3. Let M be a bound of $B_{0,3}(t)$ then for $1 \leq p < \infty$

$$\begin{aligned}
||f||_p^p &= \left\| \sum_{j=-\infty}^{\infty} x_j B_{j,3} \right\|_p^p \\
&\leq \sum_{j=-\infty}^{\infty} \int_j^{j+1} |x_j B_{j,3}(t) + x_{j+1} B_{j+1,3}(t) + x_{j+2} B_{j+2,3}(t)|^p \, dt \\
&\leq \sum_{j=-\infty}^{\infty} M^p |x_j + x_{j+1} + x_{j+2}|^p \\
&\leq 3 \cdot M^p \cdot 2^{2p-2} \sum_{j=-\infty}^{\infty} |x_j|^p = 3 \cdot 2^{2p-2} M^p ||x||_p^p.
\end{aligned}$$

The proof of the $p = \infty$ case is similar.

Next we show that there is a positive number B, independent of x and f, such that

$$||f'||_q \geq B||\Delta x||_q. \tag{3.5}$$

Since f' is the piecewise linear function joining the points $(n, (\Delta x)_n)$ we need only show, when $1 \leq q < \infty$, that for some positive number D we have

$$\int_j^{j+1} |f'(t)|^q \, dt \geq D \left(|(\Delta x)_j|^q + |(\Delta x)_{j+1}|^q \right). \tag{3.6}$$

Let $a = (\Delta x)_j$, $b = (\Delta x)_{j+1}$.

Case 1. a and b have the same sign. Suppose $a \geq 0$, $b \geq 0$. Let f_1 denote the straight line through the points (j, a) and $(j + 1/2, 0)$ and let f_2 denote the straight line through the points $(j + 1/2, 0)$ and $(j + 1, b)$. Then

$$\int_j^{j+1} |f'|^q \geq \int_j^{j+1/2} |f_1|^q + \int_{j+1/2}^{j+1} |f_2|^q = \frac{a^q}{2(q+1)} + \frac{b^q}{2(q+1)}.$$

A similar construction establishes the case when $a \leq 0$ and $b \leq 0$. Hence (3.5) holds with $B = 1/2(q+1)$ in this case.

Case 2. a and b have opposite sign. Suppose $a \geq 0$ and $b \leq 0$ and $a \geq |b|$. Let g denote the straight line through the points (j, a) and $(j + 1/2, 0)$. Then

$$\int_j^{j+1} |f'|^q \geq \int_j^{j+1/2} |g|^q = \frac{a^q}{2(q+1)} \geq \frac{a^q + |b|^q}{4(q+1)}.$$

In the last step we used $|b| \leq a$. Clearly (3.5) follows from these inequalities.

The other subcases are established similarly as is the case $q = \infty$.

Using (3.3), (3.4) and (3.5) we have that for all x in $l^p(Z)$ such that $x \neq 0$ and $0 \neq \Delta^n x$ is in $l^r(Z)$

$$\frac{||\Delta x||_q}{||x||_p^\alpha ||\Delta^2 x||_r^\beta} \leq B^{-1} \cdot A \cdot \frac{||f'||_q}{||f||_p^\alpha ||f''||_r^\beta} \leq AB^{-1} K. \tag{3.7}$$

REMARK. The proof of Theorem 3 i.e. inequality (1.1) yields an upper bound for C in terms of K:

$$C(2, 1, q, p, r; Z) \leq B^{-1} AK(2, 1, q, p, r; R). \tag{3.8}$$

Similar upper bounds for C in terms of K follow for $n > 2$ and all k, $1 \leq k < n$. However, these upper bounds for C are rough.

REFERENCES

Z. Ditzian [**1983**], *Discrete and shift Kolmogorov type inequalities*, Proc. Roy. Soc. Edinburgh **93A**, 307–313.

Z. Franco, H. G. Kaper, M. K. Kwong and A. Zettl [**1983**], *Bounds for the best constants in Landau's inequality on the line*, Proc. Roy. Soc. Edinburgh **95A**, 257–262.

V. N. Gabushin [**1967**], *Inequalities for norms of a function and its derivatives in L_p metrics*, Mat. Zametki **1**, 291–298.

H. G. Kaper and B. E. Spellman [**1987**], *Best constants in norm inequalities for the difference operator*, Trans. Amer. Math. Soc. **299**, 351–372.

M. K. Kwong and A. Zettl [**1980**], *Ramifications of Landau's inequality*, Proc. Roy. Soc. Edinburgh **86**, 175–212.

—— [**1988**], *Best constants for discrete Kolmogorov inequalities*, Houston J. Math. (to appear).

L. Nirenberg [**1955**], *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **8**, 648–674.

C. de Boor [**1978**], *A practical guide to splines*, Springer-Verlag, Berlin and New York.

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