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IDEALS OF MULTIPLIER ALGEBRAS OF SIMPLE AF C^*-ALGEBRAS

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ABSTRACT. It is shown that the C^*-algebra $M(A)/A$, where A is a nonunital separable simple AF C^*-algebra and $M(A)$ is the multiplier algebra of A, is simple if and only if A has a continuous scale or A is elementary. Some results concerning the ideal structure of $M(A)/A$ are also obtained in the case that it is nontrivial.

1. Introduction. Let K denote the C^*-algebra of all compact operators on a separable Hilbert space H, and $B(H)$ the C^*-algebra of all bounded operators on H. Then $B(H)$ is the multiplier algebra of K. (The multiplier algebra of a C^*-algebra is the idealiser of the C^*-algebra in its double dual.) It is well known that $B(H)/K$ is simple. Let A be a separable simple AF C^*-algebra with multiplier algebra $M(A)$. When is $M(A)/A$ simple? Elliott showed [**4**] that if A is an infinite, nonelementary separable matroid C^*-algebra (which is a simple AF C^*-algebra) then $M(A)/A$ has precisely one nonzero proper (closed, two sided) ideal. He also showed that if A is a finite separable matroid C^*-algebra, then $M(A)/A$ is simple. In this paper we shall consider a separable simple AF C^*-algebra A. We shall show that $M(A)/A$ is simple if and only if either A has a continuous scale or $A = K$. We shall also give some other results concerning the ideal structure of $M(A)/A$.

Recall that a separable C^*-algebra A is AF if whenever $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$ are given, there exist a finite dimensional C^*-subalgebra B of A and elements $b_1, \ldots, b_n \in B$ such that $\|a_i - b_i\| < \varepsilon$, $i = 1, 2, \ldots, n$. Furthermore, if we are initially also given a finite dimension C^*-subalgebra B_0, we may choose $B \supseteq B_0$.

Let A be a nonelementary separable simple AF C^*-algebra and G the corresponding simple dimension group with scale $\Gamma(G)$. Fix an element $u \in G^+ \backslash \{0\}$. Let $S = S_u(G)$ denote the set of all homomorphisms $\tau \colon G \to \mathbf{R}$ such that $\tau(G^+) \geq 0$ and $\tau(u) = 1$. Then S is a convex compact subset of the locally convex space $\mathbf{R}^G$ of all functions $f \colon G \to \mathbf{R}$ with the product topology. Each $\tau \in S$ can be viewed as a trace on A such that for each projection $p \in A$, $\tau(p) < \infty$. We shall denote the extreme points of S by $E(S)$. Let $\mathrm{Aff}(S)$ denote the set of all affine, real continuous functions on S. We have a positive homomorphism $\theta \colon G \to \mathrm{Aff}(S)$, $a \to \hat{a}$, where $\hat{a}(\tau) = \tau(a)$. By [**3**, Corollary 4.2], θ determines the order on G in the sense that $G^+ = \{a \in G \colon \hat{a} \gg 0\} \cup \{0\}$. Hence $G^+ \cap \ker\theta = \{0\}$. Moreover, $S = S_u(G)$ is a Choquet simplex and $H = \theta(G)$ is a dense additive subgroup of $\mathrm{Aff}(S)$. For the details of simple dimension groups readers are referred to [**3**, Chapter 4].

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For every $\tau \in S$ (as a trace), we can extend τ to a trace on $M(A)_+$. In particular, $\tau(1) = \sup\{\tau(e_n)\colon n = 1, 2, \dots\}$, where $\{e_n\}$ is an approximate identity for A consisting of projections, and 1 is the unit of $M(A)$. As in [**5**, Theorem 2], one can easily show that $\Gamma(G) = \{a \in G^+\colon \hat{a}(\tau) < \tau(1) \text{ for all } \tau \in S\}$, provided that A is nonunital.

We say that A has a continuous scale if $\hat{1}(\tau)$ is bounded and continuous on S, and a bounded scale if $\hat{1}(\tau)$ is bounded on S; we say that A is finite if $\hat{1}(\tau) < \infty$ for all $\tau \in S$. We say that A is infinite if A is not finite, and that A is stable if $\hat{1}(\tau) = \infty$ for all $\tau \in S$, which is equivalent to saying that $A \cong A \otimes K$ (see [**2**, Theorem 4.9]).

2. Simplicity of $M(A)/A$.

LEMMA 1. *Let A be a nonelementary separable infinite simple AF C^*-algebra. Let $F = \{\tau \in S\colon \hat{1}(\tau) = \infty\}$ and let J_τ be the closure of the set $\{a \in M(A)\colon \tau(a^*a) < \infty\}$ where τ is a fixed element in F. Then J_τ is an ideal of $M(A)$ such that $A \subsetneqq J_\tau \subsetneqq M(A)$.*

PROOF. Let $J_\tau^0 = \{a \in M(A)\colon \tau(a^*a) < \infty\}$. Then J_τ^0 is a *-invariant linear subspace of $M(A)$. Let $a \in J_\tau^0, b \in M(A)$. Then

$$|\tau(a^*b^*ba)| \leq \|b\|^2\tau(a^*a) < \infty.$$

Hence $ba \in J_\tau^0$; similarly $ab \in J_\tau^0$. Thus J_τ is a closed ideal of $M(A)$. Since for every projection $p \in A$, $\tau(p) < \infty$ for all $\tau \in S$, and A is AF, we conclude that $A \subseteq J_\tau$. Let $\{e_n\}$ be an approximate identity of A consisting of projections, and set $f_n = e_n - e_{n-1}$ ($e_0 = 0$). Since $\theta(G)$ is dense in $\mathrm{Aff}(S)$, there are projections $q_n \in A$ such that $0 \leq \theta[q_n] \leq 2^{-n}(1/\|\theta[f_n]\|)\theta[f_n]$, if $\|\theta[f_n]\| > 1$, or $0 < \theta[q_n] \leq 2^{-n}\theta[f_n]$, if $\|\theta[f_n]\| \leq 1$, where $\|\theta[f_n]\| = \sup\{\tau(f_n)\colon \tau \in S\}$. We may assume that $q_n \leq f_n$. Since $q_n \neq 0$ and $q_n \leq f_n$, we have that $q = \sum_{n=1}^{\infty} q_n$ is a projection in $M(A)$ but not in A. Moreover $\tau(q) = \sum_{n=1}^{\infty}\tau(q_n) \leq 1 < \infty$. So $q \in J_\tau^0 \subseteq J_\tau$. Hence $J_\tau \supsetneqq A$.

Now we show that $1 \notin J_\tau$. Otherwise there is $a \in (J_\tau^0)_+$ such that $\|1 - a\| < \frac{1}{4}$. Thus $\mathrm{sp}(a) \subset (\frac{3}{4}, \frac{5}{4})$. This implies that $0 \leq 1 \leq \frac{4}{3}a$. Then $\tau(1) < \infty$, a contradiction.

Blackadar showed in [**2**, Theorem 4.8] that A has a bounded scale if, and only if, A is algebraically simple. If A has a bounded scale, then must $M(A)/A$ be simple? We will see after the following lemma.

LEMMA 2. *Let A be a nonunital, nonelementary simple AF C^*-algebra. Let I_0 be the closure of*

$$I_{00} = \{a \in M(A)\colon \text{ there is } \{a_n\} \subset A \text{ such that } \tau((a - a_n)^*(a - a_n))$$

converges to zero uniformly on $S\}$. Then

(1) *I_0 is a (closed) ideal of $M(A)$, $A \subsetneqq I_0 \subseteq M(A)$, and I_0 is the smallest such ideal.*

(2) *If A is algebraically simple, then I_{00} is already closed.*

(3) *If A has no continuous scale, $I_0 \subsetneqq M(A)$.*

PROOF. Clearly I_{00} is a *-invariant linear subspace of $M(A)$ containing A. Suppose that $a \in M(A)$, $b \in M(A)$, and $a_n \in A$ are such that $\tau((a - a_n)^*(a - a_n)) \to 0$ uniformly on S. We have

$$\tau[(b(a - a_n))^*(b(a - a_n))] \leq \|b\|^2\tau((a - a_n)^*(a - a_n)) \to 0$$

uniformly on S. Since $ba_n \in A$, by the definition of I_{00}, $ba \in I_{00}$. Similarly $ab \in I_{00}$. Hence I_{00} is an ideal of $M(A)$. So I_0 is a closed ideal of $M(A)$. The projection q constructed in the first part of the proof of Lemma 1 is continuous on S. Moreover, we see that $\tau(\sum_{k=1}^n q_k)$ converges uniformly to $\tau(q)$ on S. Hence

$$\tau\left(\left(q-\sum_{k=1}^n q_k\right)^*\left(q-\sum_{k=1}^n q_k\right)\right)=\tau\left(q-\sum_{k=1}^n q_k\right)\to 0$$

uniformly on S. Thus $q \in I_0\backslash A$ and $I_0 \supsetneqq A$.

Let g be a projection in I_0, and let us show that $g \in I_{00}$. Since I_{00} is dense in I_0, we have that $gI_{00}g$ contains a positive element close to g, and hence contains g.

Suppose that I is another ideal such that $I \supsetneqq A$. Let $\{e_k\}$ be an approximate identity of A consisting of projections, and set $f_n = e_n - e_{n-1}$ $(e_0 = 0)$. As in the proof of [**4**, Theorem 3.1], there is a projection $p \in I\backslash A$ such that $e_k p = pe_k$. To show $I_0 \subseteq I$, it is enough to show that every projection $g \in I_0$ satisfying $e_k g = ge_k$ is in I, as in the proof of [**4**, Theorem 3.2]. Let g be such a projection. Then $g = \sum gf_n$. Also, $g \in I_{00}$, and so $\tau(g)$ is finite and continuous on S; therefore $\sum_{k=1}^n \tau(gf_k)$ converges to $\tau(g)$ uniformly on S, by Dini's theorem. We may assume that $pf_1 \neq 0$; then $\inf\{\tau(pf_1)\colon \tau \in S\} > 0$. Since $\sum_{n=1}^\infty \tau(gf_n)$ converges uniformly on S, we can choose an integer n_0 such that

$$\sum_{k\geq n_0} \tau(gf_k) < \tau(pf_1) \quad \text{for all } \tau \in S.$$

Then since infinitely many pf_n are nonzero, there exists a partition of the set $\{n_0+1, n_0+2, \dots\}$ into finite subsets $N_1, N_2, \dots$ (of consecutive integers) such that for each $n = 1, 2, \dots$, either $N_n = \varnothing$ or

$$\sum_{k\in N_n} \tau(gf_k) < \tau(pf_n) \quad \text{for all } \tau \in S.$$

Thus $[\sum_{k\in N_n} gf_k] < [pf_n]$. There exists for each $n = 1, 2, \dots, u_n \in A$ such that $u_n u_n^* = \sum_{k\in N_n} gf_k$ and $u_n^* u_n \leq pf_n$. Set $u = \sum_{n=1}^\infty u_n$. Then $u \in M(A)$, $uu^* = g - ge_{n_0}$, and $up = u$. Hence u, u^*, and g are in I. So $I_0 \subseteq I$.

Now suppose that A is algebraically simple, and let $a \in M(A)$, $b_n \in I_{00}$ be such that $\|b_n - a\| \to 0$.

We may assume that $|a - b_n| \leq 1$. Then

$$|\tau((a-b_n)^*(a-b_n))| \leq \|a-b_n\|\tau(|a-b_n|).$$

Since A has a bounded scale, $\tau(|a-b_n|) \leq \tau(1) \leq N$, for all $\tau \in S$ and some $N > 0$. Hence $\tau((a-b_n)^*(a-b_n)) \to 0$ uniformly on S. Let $a_n \in A$ be such that $\tau((b_n-a_n)^*(b_n-a_n)) < 1/n$ uniformly on S. We have

$$\tau((a-a_n)^*(a-a_n))^{1/2} \leq \tau((a-b_n)^*(a-b_n))^{1/2} + \tau((b_n-a_n)^*(b_n-a_n))^{1/2} \to 0$$

uniformly on S. We conclude that I_{00} is closed.

Finally suppose that A has no continuous scale. Then $1 \notin I_0$, i.e. $I_0 \subsetneqq M(A)$.

THEOREM 1. *Let A be a separable simple AF C^*-algebra. Then $M(A)/A$ is simple if, and only if, either A has a continuous scale or A is elementary.*

PROOF. Suppose that A is not elementary and has no continuous scale. By Lemma 2, I_0 is a closed ideal of $M(A)$ such that $A \subsetneqq I_0 \subsetneqq M(A)$. In other words, $M(A)/A$ is not simple.

If A is elementary, it is well known that $M(A)/A$ is simple. We may now assume that A has a continuou scale, i.e. that $\tau(1)$ is finite and continuous on S. By Dini's theorem, $\tau(e_n)$ converges to $\tau(1)$ uniformly on S. By the definition of I_0, $1 \in I_0$. Hence $I_0 = M(A)$. By Lemma 2, I_0 is the smallest ideal containing A. We conclude that $M(A)/A$ is simple.

REMARKS. Theorem 1 implies Theorem 3.1 of [**4**].

Given a simple dimension group G we can construct a separable, nonunital, simple AF C^*-algebra A with a continuous scale such that the dimension group of A is G. So for every separable, nonunital simple AF C^*-algebra A, there is a separable, nonunital simple AF C^*-algebra B such that $A \otimes K \cong B \otimes K$ and $M(B)/B$ is simple.

3. Ideals of $M(A)/A$. Let A be a nonunital, separable, simple AF C^*-algebra, and let G and $S = S_u(G)$ be as before. Set $F = \{\tau \in S \colon \tau(1) = \infty\}$ and let α be a subset of $F \cap E(S)$. Let I_α denote the closure of the set $\{a \in M(A) \colon \tau(a^*a) < \infty$ for all $\tau \in \alpha\}$. Then, as is easily seen, I_α is an ideal of $M(A)$ containing A. The following theorem is a generalization of Theorem 3.2 of [**4**].

THEOREM 2. *Let A be a nonunital, nonelementary, separable simple AF C^*-algebra. Suppose that $E(S)$ has only finitely many points and $F \cap E(S)$ has n points. Then $M(A)/A$ has exactly $2^n - 1$ different proper closed ideals, each of which has the form I_α/A.*

PROOF. Suppose that $n = 0$. Since $E(S)$ has finitely many points, A has a continuous scale. In this case, Theorem 2 follows from Theorem 1.

We now suppose that $F \cap E(S) = \{\tau_1, \ldots, \tau_n\}$, $n \geq 1$. As in the proof of Lemma 1, each I_α is a proper closed ideal of $M(A)$ containing A properly. Let us show that if α, β are nonempty subsets of $F \cap E(S)$ with $\alpha \neq \beta$, then $I_\alpha \neq I_\beta$. We may assume that $\alpha = \{\tau_1, \ldots, \tau_k\}$ where $k < n$, and that $\tau_{k+1} \in \beta$. For each $n = 1, 2, \ldots$, let $h_n \in \mathrm{Aff}(S)$ be such that

$$\theta[f_n](\tau_{k+1}) > h_n(\tau_{k+1}) > \tfrac{1}{2}\theta[f_n](\tau_{k+1}),$$

and

$$0 < h_n(\tau_i) \leq \min(2^{-n}, \theta[f_n](\tau_i)), \qquad i = 1, 2, \ldots, k.$$

(Since $E(S)$ is finite, the existence of h_n is clear.) Since $\theta(G)$ is dense in $\mathrm{Aff}(S)$, we may assume that $h_n \in \theta(G)$. So we have projections $p_n \in A$ such that $p_n \leq f_n$ and $\tau_i(p_n) \leq 2^{-n}$, $i = 1, 2, \ldots, k$, $\tau_{k+1}(p_n) \geq \frac{1}{2}\tau_{k+1}(f_n)$. Then with $p = \sum p_n$, we have $p \in M(A)$ and $p \in I_\alpha$ but $p \notin I_\beta$; this is proved in the same way as $1 \notin J_\tau$ in Lemma 1. Thus $I_\alpha \neq I_\beta$.

Suppose that I is a closed ideal of $M(A)$. We shall show that I is equal to the smallest I_α which contains it (α could be the empty set).

Let I_α be such an ideal. Write $F \cap (E(S)\backslash\alpha) = \{\tau_1, \tau_2, \ldots, \tau_s\}$, and set $\alpha_i = \alpha \cup \{\tau_i\}$. Since $I \not\subset I_{\alpha_i}$, there are projections $g_i \in I \backslash I_{\alpha_i}$ such that $f_k g_i = g_i f_k$ for $i = 1, 2, \ldots, s$ and $k = 1, 2, \ldots$ (see the proof of [**4**, Theorem 3.2]). Thus $\tau_i(g_i) = \infty$.

Changing $g_i f_k$ into equivalent projections, we may assume that they belong to a common finite dimensional C^*-subalgebra of $f_k A f_k$, say B_k. Then the range projection h_k of $(\sum_{i=1}^s g_i) f_k$ exists in B_k. Since B_k is a finite dimensional C^*-algebra, $\{(\sum_{i=1}^s g_i) f_k\}^{1/n} \to h_k$ in norm. Hence $(\sum_{i=1}^s g_i) f_k$ has an inverse b_k in the C^*-subalgebra $h_k B_k h_k$. Set $h = \sum_{k=1}^\infty h_k$ and $b = \sum_{k=1}^\infty b_k$. Both h and b are in $M(A)$. Since $h = b(\sum_{i=1}^{s} g_i)$, $h \in I$. Clearly $\tau(h) \geq \tau(g_i)$, $i = 1, 2, \ldots, s$. So $\tau_i(h) = \infty$ for $i = 1, 2, \ldots, s$.

As in the proof of [**4**, Theorem 3.27], to show that $I \supset I_\alpha$, it is enough to show that every projection $q \in I_\alpha$ such that $f_k q = q f_k$ is in I. Suppose that q is such a projection. There exists a partition of $\{1, 2, \ldots\}$ into finite sets $N_1, N_2, \ldots$ (of consecutive integers) such that for each $m = 1, 2, \ldots,$

$$\tau_i(qf_m) < \sum_{k \in N_m} \tau_i(h_k), \qquad i = 1, 2, \ldots, s.$$

Let β_m denote the set of τ in $E(S)$ such that

$$\tau(qf_m) > \sum_{k \in N_m} \tau(h_k).$$

Then $\beta_m \subset \alpha \cup [E(S) \backslash F]$. Since $\theta(G)$ is dense in $\mathrm{Aff}(S)$, for each m, there is a projection $q_m < qf_m$ such that

$$0 < \tau(qf_m - q_m) < \sum_{k \in N_m} \tau(h_k)$$

for $\tau \in \beta_m$ and

$$0 < \tau(q_m) < 1/2^m \quad \text{for } \tau \in E(S) \backslash \beta_m.$$

Thus $q_0 = \sum_{m=1}^{\infty} q_m$ is in I_0, the closure of $\{a \in M(A)\colon \tau(a^*a) < \infty$ for all $\tau \in E(S)\}$. Set $q' = q - q_0$; then

$$\tau(q' f_m) < \sum_{k \in N_m} \tau(h_k)$$

for all $\tau \in E(S)$, hence for all $\tau \in S$. Therefore there exists for each $m = 1, 2, \ldots,$ $v_m \in A$ such that $v_m v_m^* = q' f_m$ and $v_m^* v_m \leq \sum_{k \in N_m} h_k$. Set $v = \sum_{m=1}^{\infty} v_m$; then $v \in M(A)$, and $q' f_m v = v \sum_{k \in N_m} h_k = v_m$. In particular v is a partial isometry, and $vv^* = q' = q - q_0$ and $vh = v$. Then v, v^*, and therefore $q - q_0$ are in I. Since $E(S)$ is finite, by Lemma 2, I_0 is the smallest ideal in $M(A)$ properly containing A. So $I \supseteq I_0$, whence $q_0 \in I$ and $q \in I$. This completes the proof.

THEOREM 3. *Let A be a nonelementary separable infinite simple AF C^*-algebra. Suppose that $F \cap E(S)$ is an infinite set. Then $M(A)/A$ has infinitely many different (closed) ideals.*

PROOF. Let $\{\tau_i\}$ be a sequence in $F \cap E(S)$. Let $F_k = \{\tau_i\colon i = 1, 2, \ldots, k+1\}$ and let J_k be the closure of $\{a \in M(A)\colon \tau_i(a^*a) < \infty, i = 1, 2, \ldots, k+1\}$. Define

$$h_n(\tau_i) = \min\{2^{-n-1}, \theta[f_n](\tau)\}, \qquad i = 1, 2, \ldots, k,$$

and

$$h_n(\tau_{k+1}) = \tfrac{1}{2}\theta[f_n](\tau_{k+1}).$$

Define $\underline{h}_n(\tau) = h_n(\tau) = \overline{h}_n(\tau)$ for $\tau \in F_k$, $\underline{h}_n(t) = \inf\{h_n(\tau)\colon \tau \in F_k\}$, $\overline{h}_n(t) = \sup\{h_n(\tau)\colon \tau \in F_k\}$, for $t \in S \backslash F_k$. It is easily verified that $\underline{h}_n$ is upper semicontinuous and convex while $\overline{h}_n$ is lower semicontinuous and concave. By [**1**, Theorem II.3.10] there exists a real affine continuous function g_n' on S such that

$0 < \underline{h}_n \le g'_n \le \overline{h}_n$. Hence $g'_n|F_k = h_n$. Since $\theta(G)$ is dense in $\mathrm{Aff}(S)$, for each n there is $g_n \in \theta(G)$ such that

$$|g_n(t) - \tfrac{1}{2}g'_n(t)| < \tfrac{1}{4}\inf\{h_n(\tau) : \tau \in F_k\}$$

for all $t \in S$. Consequently, we have projections $p_n \in A$ such that $p_n \le f_n$, $\tau_i(p_n) \le 2^{-n}$, $i = 1, 2, \ldots, k$ and $\tau_{k+1}(p_n) \ge \frac{1}{8}\tau_{k+1}(f_n)$. Set $p = \sum_{n=1}^{\infty} p_n$. Then $p \in M(A)$. It is easily verified that $p \in J_k$ but $p \notin J_{k+1}$ (just as $1 \notin J_\tau$ in Lemma 1). Thus $J_k \supsetneqq J_{k+1}$. This completes the proof.

REMARK. Let A be a nonunital, nonelementary, separable simple AF C^*-algebra without continuous scale, such that $E(S)$ is infinite. If furthermore, $E(S)$ is closed or, equivalently, S is a Bauer simplex, then every real continuous function on $E(S)$ can be extended to a function in $\mathrm{Aff}(S)$. Therefore an argument similar to that used in this paper shows that $M(A)/A$ has infinitely many closed ideals. We believe that $M(A)/A$ has infinitely many closed ideals even if $E(S)$ is not closed. However, if S is a general Choquet simplex, a continuous function on $E(S)$ may not extend to a continuous affine function on S, and this creates a technical problem. Other methods may be needed.

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REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Math., vol. 57, Springer-Verlag, Berlin and New York, 1971.
2. B. E. Blackadar, *Traces on simple AF C^*-algebras*, J. Funct. Anal. **38** (1980), 156–168.
3. E. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math., no. 46, Amer. Math. Soc., Providence, R.I., 1981, pp. 1–74.
4. G. A. Elliott, *Derivations of matroid C^*-algebras*. II, Ann. of Math. (2) **100** (1974), 407–422.
5. H. Lin, *Fundamental approximate identities and quasi-multipliers of simple AF C^*-algebras*, J. Funct. Anal. (to appear).

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