PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 1, September 1988

A DE RHAM TYPE THEOREM FOR ORBIT SPACES

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(Communicated by Haynes R. Miller)

ABSTRACT. Let G be a compact Lie group and M be a smooth G-space. We prove that the real cohomology algebra of the orbit space M/G is isomorphic to the homology algebra of the de Rham complex of G-basic differential forms on M.

Let G be a compact Lie group and M be a smooth G-manifold, i.e. M is a paracompact, C^∞-manifold and there is given a C^∞-map $f\colon G\times M \to M$, $f(g,x) = gx$, such that (i) $g(hx) = (gh)x$ for $g,h \in G$, $x \in M$; and (ii) $ex = x$ for $x \in M$, $e \in G$ being the neutral element (see **[2]** for all other notions and results concerning G-manifolds that will be used in this note). For $g \in G$ let $L_g\colon M \to M$ denote the diffeomorphism $L_g(x) = gx$. For $x \in M$ let $G_x = \{g \in G; gx = x\}$ be the isotropy subgroup at x and $Gx = \{gx; g \in G\}$ be the G-orbit through x. Let M/G denote the orbit space and $p\colon M \to M/G$ be the canonical projection.

Let Lie(G) be the Lie algebra of G and for $X \in \mathrm{Lie}(G)$ let X^* be the induced vector field on M: for each $x \in M$, $X^*(x)$ is the vector tangent to the curve $t \mapsto (\exp(tX))x$ at $t = 0$.

For $r \geq 0$ let $\mathscr{D}^r(M)$ denote the (real) vector space of C^∞-forms on M of degree r. Given $X \in \mathrm{Lie}(G)$ let $i_X\colon \mathscr{D}^r(M) \to \mathscr{D}^{r-1}(M)$ be the interior product with X^*. Let also $d\colon \mathscr{D}^r(M) \to \mathscr{D}^{r+1}(M)$ denote the exterior derivative. A form $\omega \in \mathscr{D}^r(M)$ is called G-basic if

$$(L_g)^*(\omega) = \omega \quad \text{and} \quad i_X(\omega) = 0 \qquad \text{for any } g \in G, X \in \mathrm{Lie}(G).$$

We shall denote by $\mathscr{D}^r_G(M)$ the real vector space of G-basic forms on M of degree r. It is easy to check that $d(\mathscr{D}^r_G(M)) \subset \mathscr{D}^{r+1}_G(M)$ and therefore we can consider the quotient $H^r_G(M) = (\ker(d) \cap \mathscr{D}^r_G(M))/d(\mathscr{D}^{r-1}_G(M))$. Let $H^*_G = \bigoplus_{r\geq 0} H^r(M)$ and notice that the exterior product determines an algebra structure on $H^*_G(M)$.

THEOREM. *The algebra $H^*_G(M)$ is isomorphic to the real cohomology algebra of M/G.*

PROOF. For $U \subset M/G$ open and $r \geq 0$, let $S^r(U) = \mathscr{D}^r_G(p^{-1}(U))$. If $V \subset U$ is also open, there exists an obvious restriction homomorphism $S^r(U) \to S^r(V)$. The assignment $U \mapsto S^r(U)$ is a sheaf over M/G; denote it $\mathbf{S}^r$. The exterior derivative gives rise to sheaf morphisms $d\colon \mathbf{S}^r \to \mathbf{S}^{r+1}$, $r \geq 0$. The Theorem will follow if we can prove that

$$(*) \qquad 0 \to \mathbf{R} \to \mathbf{S}^0 \to \mathbf{S}^1 \to \mathbf{S}^2 \to \cdots$$

Received by the editors April 1, 1987 and, in revised form, August 20, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57S15, 58A12.
Research partially supported by NSF Grant DMS-8509186.

is a fine resolution of $\mathbf{R}$, the constant sheaf over M/G with stalk R (the reals). Since there are partitions of unity on M consisting of G-invariant C^∞-functions, $\mathbf{S}^0$ is fine and therefore so is each $\mathbf{S}^r$, $r \geq 1$.

Let $x \in M$, $H = G_x$, and $N = T_xM/T_x(Gx)$ be the normal space at x to the orbit Gx. The action of G on M induces a linear action of H on N (the so-called slice (or isotropy) representation), and there exists an open, G-invariant neighborhood U of x in M which is G-equivariantly diffeomorphic to the twisted product $G \times_H N$ on which G acts by left multiplication (this is the "slice theorem"; recall that $G \times_H N$ is the orbit space $(G \times N)/H$, where H acts on $G \times N$ by $h(g, v) = (gh^{-1}, hv)$). Consider the commutative diagram

$$\begin{tikzcd}
S^{r-1}(U/G) \arrow[r, "d"] \arrow[d, "="'] & S^r(U/G) \arrow[r, "d"] \arrow[d, "="'] & S^{r+1}(U/G) \arrow[d, "="'] \\
\mathscr{D}_G^{r-1}(U) \arrow[r, "d"] \arrow[d] & \mathscr{D}_G^r(U) \arrow[r, "d"] \arrow[d] & \mathscr{D}_G^{r+1}(U) \arrow[d] \\
\mathscr{D}_G^{r-1}(G \times_H N) \arrow[r, "d"] \arrow[d] & \mathscr{D}_G^r(G \times_H N) \arrow[r, "d"] \arrow[d] & \mathscr{D}_G^{r+1}(G \times_H N) \arrow[d] \\
\mathscr{D}_H^{r-1}(N) \arrow[r, "d"] & \mathscr{D}_H^r(N) \arrow[r, "d"] & \mathscr{D}_H^{r+1}(N)
\end{tikzcd}$$

in which the second row of vertical maps is induced by the above-mentioned diffeomorphism and the third one by the map $x \mapsto [e, x] \colon N \to G \times_H N$ (here $[e, x]$ denotes the H-orbit through $(e, x) \in G \times N$). It is easy to see that all vertical maps are isomorphisms. Let us check that the bottom row is exact. To this end consider the homotopy operator $A \colon \mathscr{D}^r(N) \to \mathscr{D}^{r-1}(N)$ (see, for example, [**4**, p. 128]). A direct computation shows that for $\omega \in \mathscr{D}^r(N)$, $h \in H$ and $X \in \mathrm{Lie}(H)$

$$A((L_h)^*(\omega)) = (L_h)^*(A(\omega)) \quad \text{and} \quad A(i_X(\omega)) = i_X(A(\omega))$$

(the first equality is true for any C^∞-map $f \colon N \to N$ such that $f(tx) = tf(x)$, $t \in R$, $x \in N$; the second equality is true for any vector field Y on N such that $Y(tx) = tY(x)$, $t \in R$, $x \in N$, where $T_{tx}(N)$ and $T_x(N)$ are identified as usually to N).

It follows that $A(\mathscr{D}_H^r(N)) \subset \mathscr{D}_H^{r-1}(N)$. Thus if $\omega \in \mathscr{D}_H^r(N)$ is such that $d\omega = 0$, then $\omega = d\theta$ with $\theta = A\omega \in \mathscr{D}_H^{r-1}(N)$. This proves the exactity of the bottow row. Therefore, the first row is also exact.

Since the open sets of the form U/G with U as above form a fundamental system of neighborhoods of $p(x)$ in M/G, it follows that $(*)$ is a resolution of $\mathbf{R}$. The Theorem follows from Theorem 4.7.1 in [**3**].

Note. The particular case of the Theorem when all the isotropy groups are finite was proved in [**1**, Lemma 10.2] by other methods.

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