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TWO NEW TOPOLOGICAL CARDINAL INEQUALITIES

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ABSTRACT. In this paper, two new cardinal inequalities are obtained. The first one is a common generalization of inequalities of Hajnal-Juhasz and Sapirovskii; the second one generalizes the inequality of Arhangelskiĭ.

It is well known that Hajnal-Juhasz's inequality [**3**], "For $X \in T_2$, $|X| \leq 2^{c(X)\chi(X)}$" and Sapirovskii's inequality [**2**], "For $X \in T_3$, $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$" are still two of the best cardinal inequalities. In other words, they have not been improved so far [**2**].

In the first part of this paper, we will establish a stronger inequality, "For $X \in T_2$, $|X| \leq \pi\chi(X)^{c(X)S\psi(X)}$" which is the common generalization of the two inequalities above. Here, we define a collection $\mathscr{U}$ of open sets of X to be a *strong pseudo-local base* at p if $\{p\} = \bigcap\{U : U \in \mathscr{U}\} = \bigcap\{\overline{U} : U \in \mathscr{U}\}$. Then we define:

$$S\psi(p, X) = \min\{|\mathscr{U}| : \mathscr{U} \text{ is a strong pseudo-local base at } p\} \cdot \omega,$$
$$S\psi(X) = \sup\{S\psi(p, X) : p \in X\}.$$

It is immediate that $\psi(X) = S\psi(X)$ for T_3 spaces and that $\chi(X) \geq S\psi(X)$ for T_2 spaces. The latter inequality can be strict; for example, consider the subspace $N \cup \{p\}$ of βN, where $p \in \beta N \backslash N$.

For each space X it is obvious that $\pi\chi(X) \leq \chi(X) < 2^{c(X)\chi(X)}$. Thus

$$\pi\chi(X)^{c(X)S\psi(X)} \leq (2^{c(X)\chi(X)})^{c(X)S\psi(X)} \leq (2^{c(X)\chi(X)})^{c(X)\chi(X)} = 2^{c(X)\chi(X)}.$$

Hence our new inequality is at least as strong as that of Hajnal and Juhasz. Later we give an example to show that the above inequality can be strict. Because $\psi(X) = S\psi(X)$ when X is T_3, it follows that our inequality also generalizes that of Sapirovskii mentioned above.

Our proofs make use of the Pol-Sapirovskii technique. We shall use the notation and terminology for cardinal functions employed in [**3**]. For the convenience of the reader, we repeat some of the definitions contained in this paper.

First, let $|A|$ denote the cardinality of A; let λ, k be infinite cardinals and let ω denote the smallest infinite ordinal and the smallest infinite cardinal. The successor of k will be denoted by k^+.

DEFINITIONS. Let X be a topological space and $\mathscr{U}$ be a collection of nonempty open sets of X. Let $p \in X$; then $\mathscr{U}$ is a local π-base at p if for each neighbourhood

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R of p, there exists a $V \in \mathscr{U}$ such that $V \subseteq R$. We define

$$\begin{aligned}\pi\chi(p, X) &= \min\{|\mathscr{U}|: \mathscr{U} \text{ is a local } \pi\text{-base for } p\} \cdot \omega;\\ \pi\chi(X) &= \sup\{\pi\chi(p, X): p \in X\};\\ \chi(p, X) &= \min\{|\mathscr{U}|: \mathscr{U} \text{ is a local base for } p\} \cdot \omega;\\ \chi(X) &= \sup\{\chi(p, X): p \in X\}.\end{aligned}$$

A collection $\mathscr{U}$ of open sets in X is said to be a pseudo-local base for p, if $\{p\} = \bigcap\{U: U \in \mathscr{U}\}$. Now we define

$$\begin{aligned}\psi(p, X) &= \min\{|\mathscr{U}|: \mathscr{U} \text{ is a pseudo-local base for } p\} \cdot \omega;\\ \psi(X) &= \sup\{\psi(p, X): p \in X\}.\end{aligned}$$

A pairwise disjoint collection of nonempty open sets of X is said to be a cellular family. We define $c(X) = \sup\{|\mathscr{U}|: \mathscr{U} \text{ is a cellular family in } X\}$.

The notation $[A]^{\leq k}$ denotes the set $\{B: B \subseteq A, |B| \leq k\}$.

THEOREM 1. *For $X \in T_2$, $|X| \leq \pi\chi(X)^{c(X)S\psi(X)}$.*

PROOF. Let $\pi\chi(X) = \lambda$ and $c(X)S\psi(X) = k$; for each p in X, let $\mathscr{U}_p$ be a π-base for p such that $|\mathscr{U}_p| \leq \lambda$.

Construct a sequence $\{A_\alpha: \alpha < k^+\}$ of subsets of X and a sequence $\{\mathscr{U}_\alpha: 0 < \alpha < k^+\}$ of open collections in X such that

(1) $|A_\alpha| \leq \lambda^k$, $0 \leq \alpha < k^+$;

(2) $\mathscr{U}_\alpha = \{V: \exists p \in \bigcup_{\beta<\alpha} A_\beta \text{ such that } V \in \mathscr{U}_p\}$, $0 < \alpha < k^+$;

(3) For each $r < k$, if $\mathscr{V}_r \in [\mathscr{U}_\alpha]^{\leq k}$ and $W = \bigcup_{r<k} \overline{\bigcup \mathscr{V}_r} \neq X$, then $A_\alpha \backslash W \neq \varnothing$.

The construction is by transfinite induction. Let $0 < \alpha < k^+$ and assume that $\{A_\beta: \beta < \alpha\}$ has already been constructed. Then $\mathscr{U}_\alpha$ is defined by (2), i.e., we put $\mathscr{U}_\alpha = \{V: \exists p \in \bigcup_{\beta<\alpha} A_\beta, V \in \mathscr{U}_p\}$. It follows that $|\mathscr{U}_\alpha| \leq \lambda^k$. If $\{\mathscr{V}_r\}_{r<k} \in [[\mathscr{U}_\alpha]^{\leq k}]^{\leq k}$ and $W = \bigcup_{r<k} \overline{\bigcup \mathscr{V}_r} \neq X$, then we can choose one point of $X \backslash W$. Let S_α be the set of points chosen in this way. Since $|[[\mathscr{U}_\alpha]^{\leq k}]^{\leq k}| \leq \lambda^k$, one has $|S_\alpha| \leq \lambda^k$. Define A_α to be the set $S_\alpha \cup (\bigcup_{\beta<\alpha} A_\beta)$. Then A_α satisfies (1), and (3) is also satisfied if $\beta \leq \alpha$. This completes the construction.

Now let $S = \bigcup_{\alpha<k^+} A_\alpha$; then $|S| \leq k^+\lambda^k = \lambda^k$. The proof is complete if $S = X$. Suppose not and let $p \in X \backslash S$; since $S\psi(X) \leq k$, there exist open neighbourhoods $\{U_\alpha(p)\}_{\alpha<k}$ of p such that $\{p\} = \bigcap_{\alpha<k}\{\overline{U_\alpha(p)}\}$. For each $\alpha < k$, let $V_\alpha = X \backslash \overline{U_\alpha(p)}$, then $p \notin \overline{V}_\alpha$, and $\bigcup_{\alpha<k} V_\alpha = X \backslash \{p\} \supset S$.

For each $\alpha < k$, and for each $q \in (V_\alpha \cap S) \subseteq V_\alpha$, there exists a $V_q \in \mathscr{U}_q$ such that $V_q \subseteq V_\alpha$. Let $\mathscr{W}_\alpha = \{V: V \in \mathscr{U}_q, q \in V_\alpha \cap S, V \subseteq V_\alpha\}$; then $G_\alpha = \bigcup \mathscr{W}_\alpha \subseteq V_\alpha$, and it is easy to check that $V_\alpha \cap S \subseteq G_\alpha$. As $c(X) \leq k$, there exists a $\mathscr{V}_\alpha \in [\mathscr{W}_\alpha]^{\leq k}$ such that $\overline{\bigcup \mathscr{V}_\alpha} = \overline{G_\alpha} \supseteq V_\alpha \cap S$, and $p \notin \overline{\bigcup \mathscr{V}_\alpha}$. Let $W = \bigcup_{\alpha<k} \overline{\bigcup \mathscr{V}_\alpha}$ then $p \notin W$.

Since $|\{V: \exists \alpha < k \text{ such that } V \in \mathscr{V}_\alpha\}| \leq kk = k < k^+$, there is an $\alpha_0 < k^+$ such that $\mathscr{V}_\alpha \in [\mathscr{U}_{\alpha_0}]^{\leq k}$ for each $\alpha < k$. Hence, by (3), one has $A_{\alpha_0+1} \backslash W \neq \varnothing$. But $W \supseteq \bigcup_{\alpha<k}(V_\alpha \cap S) = S$. This is a contradiction.

COROLLARY 1 (SAPIROVSKII). *For $X \in T_3$, $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$.*

PROOF. It is clear that $\psi(X) = S\psi(X)$ for $X \in T_3$.

COROLLARY 2 (HAJNAL-JUHASZ). *For $X \in T_2$, $|X| \leq 2^{c(X)\chi(X)}$.*

PROOF. By this theorem, $|X| \leq \pi\chi(X)^{c(X)S\psi(X)} \leq 2^{\pi\chi(X)c(X)S\psi(X)} \leq 2^{c(X)\chi(X)}$.

REMARK 1. In this theorem, we omitted the condition "regularity". This improvement on Sapirovskii's theorem is not trivial, as there exist T_2 spaces for which Sapirovskii's inequality fails. For example, the Katetov H-closed extension of the positive integers, denoted kN, is such a space as $|kN| = 2^c$, $\pi\chi(kN) = \omega$, and $\psi(kN) = \omega$.

REMARK 2. Our theorem represents a considerable improvement on the theorem of Hajnal and Juhasz quoted above, as the gap between $\pi\chi(X)^{c(X)S\psi(X)}$ and $2^{c(X)\chi(X)}$ can be large.

To illustrate this, let us consider the Cantor cube $D^c = \prod_{s\in S} D_s$, where $D_s = D$, D denotes the two-point discrete space, for every $s \in S$, and $|S| = c$. It contains a dense countable subspace X such that $\chi(X) = c$ [**1**] and it is clear that $\psi(X) = c(X) = |X| = \omega$. By the regularity of X, one has $S\psi(X) = \omega$. Note that $\pi\chi(X) \leq \chi(X) = c = 2^\omega$. So $\pi\chi(X)^{c(X)S\psi(X)} \leq (2^\omega)^{\omega\omega} = 2^\omega = c$. But $2^{c(X)\chi(X)} = 2^c$.

We also note that if $p \in \beta N \backslash N$, the space $N \cup \{p\}$ illustrates the same point.

Now we turn to the second result of this paper. We shall establish a theorem which strengthens Arhangelskii's theorem, "For $X \in T_2$, $|X| \leq 2^{t(X)L(X)\psi(X)}$" (see [**2**, Remark 4.6]), where $t(X)$ and $L(X)$ denote the tightness and Lindelöf degree of X, respectively.

Let k be a cardinal number. We define a subset A of X with $|A| \leq 2^k$ to be *k-quasi-dense* if for each open cover $\mathscr{U}$ of X there exist a $\mathscr{V} \in [\mathscr{U}]^{\leq k}$ and a $B \in [A]^{\leq k}$ such that $(\bigcup \mathscr{V}) \cup \overline{B} = X$; and let us write $qL(X) = \min\{k\colon$ *there is a k-quasi-dense subset A*$\}$. (The notion of $qL(X)$ was first considered by Liu Xiao-Shi.) Clearly, $qL(X)$ is less than both the density $d(X)$ and Lindelöf degree $L(X)$. We also can prove that $qL(X) \leq s(X)$, where

$$s(X) \quad (= \sup\{|D| \colon D \subseteq X, D \text{ is discrete}\}\omega)$$

denotes the spread of X. In fact, by virtue of the theorem of Sapirovskii, "Let $X \in T_2$, let $s(X) \leq k$. Then there is a subset S of X with $|S| \leq 2^k$ such that $X = \bigcup\{\overline{A}\colon A \subseteq S, |A| \leq k\}$", we have only to show that the subset S is a k-quasi-dense subset of X. Let $\mathscr{U}$ be an open cover of X. By virtue of another theorem of Sapirovskii which asserts that if $\mathscr{U}$ is an open cover of X with $s(X) \leq k$, then there is a subset B of X with $|B| \leq k$ and a subcollection $\mathscr{V}$ of $\mathscr{U}$ with $|\mathscr{V}| \leq k$ such that $X = \overline{B} \cup (\bigcup \mathscr{V})$ [**2**, Proposition 4.8], we can easily find a subset A of S with $|A| \leq k$ such that $X = \overline{A} \cup (\bigcup \mathscr{V})$. In fact, for each $b \in B$, there is a subset $A(b)$ of S with $|A(b)| \leq k$ such that $b \in \overline{A(b)}$. Let $A = \bigcup\{A(b)\colon b \in B\}$; then A is as required.

LEMMA. *Let X be a space with $t(X)S\psi(X) \leq k$. Then for each subset A of X with $|A| \leq 2^k$, we have $|\overline{A}| \leq 2^k$.*

PROOF. Let $x \in \overline{A}$; since $S\psi(X) \leq k$, there exists a strong pseudo-local base $\{U_\alpha(X)\colon \alpha < k\}$ at x. Thus, $\{x\} = \bigcap_{\alpha<k} \overline{U_\alpha(x) \cap A}$. Since $t(X) \leq k$, for each $\alpha < k$, there exists $A_\alpha \subseteq U_\alpha(x) \cap A \subseteq A$ such that $|A_\alpha| \leq k$ and $x \in \overline{A}_\alpha$. Hence $\{x\} = \bigcap_{\alpha<k} \overline{A}_\alpha$ and $\{A_\alpha\}_{\alpha<k} \in [[A]^{\leq k}]^{\leq k}$. Therefore $|\overline{A}| \leq |[[A]^{\leq k}]^{\leq k}| = 2^k$.

REMARK. The above lemma generalizes Pospisil's inequality, which states that if $X \in T_2$, then $|X| \leq d(X)^{\chi(X)}$ [**2**, Theorem 4.4].

THEOREM 2. *For $X \in T_2$, $|X| \leq 2^{qL(X)t(X)S\psi(X)}$.*

PROOF. Let $qL(X)t(X)S\psi(X) = k$; and let A be a k-quasi-dense subset of X. For each $x \in X$, let $\mathscr{V}_x$ denote a family of open neighbourhoods $\{U_\alpha(x)\}_{\alpha<k}$ such that $\{x\} = \bigcap_{\alpha<k} \overline{U_\alpha(x)}$ and $|\mathscr{V}_x| \leq k$. Using the above lemma, we construct an increasing sequence $\{H_\alpha\colon 0 \leq \alpha < k^+\}$ of closed sets of X and a sequence $\{\mathscr{V}_\alpha\colon 0 < \alpha < k^+\}$ of open collections in X such that:

(1) $|H_\alpha| \leq 2^k$, $0 \leq \alpha < k^+$;

(2) $\mathscr{V}_\alpha = \{V : V \in \mathscr{V}_p, p \in \bigcup_{\beta<\alpha} H_\beta\}$, $0 < \alpha < k^+$;

(3) if W is the union of $\leq k$ elements of $\mathscr{V}_\alpha$, $B \in [A]^{\leq k}$ and $X \backslash (W \cup \overline{B}) \neq \varnothing$, then $H_\alpha \backslash (W \cup \overline{B}) \neq \varnothing$.

Now let $H = \bigcup_{\alpha<k^+} H_\alpha$; as $t(X) \leq k$ and H_α is closed for each $\alpha < k^+$, H is closed. We will show that $H \cup \overline{A} = X$, from which it follows that $|X| \leq |H| + |\overline{A}| \leq 2^k$.

Let $q \in X \backslash H$. For each $p \in H$, choose a $V_p \in \mathscr{V}_p$ such that $q \notin \overline{V}_p$. Now $\bigcup\{V_p : p \in H\} \supseteq H$. Since $qL(X) \leq k$, there exist a $\mathscr{V} \in [\{V_p : p \in H\}]^{\leq k}$ and a $B \in [A]^{\leq k}$ such that $H \subseteq (\bigcup \mathscr{V}) \cup \overline{B}$. If $q \in \overline{B} \subseteq \overline{A}$, then the conclusion holds. If $q \notin \overline{B}$, then $q \notin (\bigcup \mathscr{V}) \cup \overline{B}$, i.e., $(\bigcup \mathscr{V}) \cup \overline{B} \neq X$. Hence we can find a $\beta < k^+$ with $\mathscr{V} \in [\mathscr{V}_\beta]^{\leq k}$ since $|\mathscr{V}| \leq k$ and $H = \bigcup_{\alpha<k^+} H_\alpha$. So $H_\beta \backslash ((\bigcup \mathscr{V}) \cup \overline{B}) \neq \varnothing$, which contradicts with the fact that $H \subseteq (\bigcup \mathscr{V}) \cup \overline{B}$.

COROLLARY. *For $X \in T_2$, we have $|X| \leq 2^{L(X)t(X)\psi(X)}$.*

PROOF. It is easily checked that $S\psi(X) \leq L(X)\psi(X)$, so

$$L(X)\psi(X) = L(X)S\psi(X).$$

The conclusion now follows from the theorem above.

EXAMPLE. Let X be the Niemytzki plane. Then $d(X) = qL(X) = \chi(X) = t(X) = S\psi(X) = \omega$, but $L(X)t(X)\psi(X) \geq L(X) > \omega$.

REMARK. In Theorem 2, $S\psi(X)$ cannot be replaced by $\psi(X)$ since there exists a Hausdorff space X of cardinality 2^c (namely the Katetov H-closed extension of N) that contains a countable dense subset A consisting of isolated points of X such that the subspace $X \backslash A$ is discrete. It is easy to check $qL(X) = d(X) = \psi(X) = t(X)$, but $|X| > c = 2^{qL(X)t(X)\psi(X)}$.

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