PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

A NOTE ON GENERALIZED INVERSE FUNCTIONS

ROBIN HARTE

(Communicated by Paul S. Muhly)

ABSTRACT. We offer the following generalization of an observation of Graham Allen about left inverses: if the elements of a semigroup of Banach algebra elements can all be given mutually commuting generalized inverses, then they can all be given invertible generalized inverses.

If A is a ring, with identity 1, we shall write A^{-1} for the invertible group and

$$(0.1)\qquad A^{-1}_{\text{left}} = \{a \in A \colon 1 \in Aa\}, \qquad A^{-1}_{\text{right}} = \{a \in A \colon 1 \in aA\}$$

for the larger semigroups of left and of right invertible elements, whose intersection is of course A^{-1}; more generally [**2**, **3**] we write

$$(0.2)\qquad \overline{A} = \{a \in A \colon a \in aAa\}$$

for the subset of those elements of A which have *generalized inverses*. The *commutant* of a subset $K \subseteq A$ is the set

$$(0.3)\qquad \operatorname{comm}_A(K) = \{a \in A \colon ab = ba \textit{ for each } b \in K\};$$

in particular, the *centre* of A is given by

$$(0.4)\qquad \operatorname{Centre}(A) = \operatorname{comm}_A(A).$$

We shall describe the ring A as *centre-commutative* if

$$(0.5)\qquad \textit{for each } a \in A \textit{ there is } \lambda \in \operatorname{Centre}(A) \textit{ for which } a - \lambda \in A^{-1}.$$

This is familiar for example if A is a real or a complex Banach algebra:

$$(0.6)\qquad \lambda \textit{ scalar and } |\lambda| > ||a|| \Rightarrow \exists (a-\lambda)^{-1} = -\lambda^{-1}\left(1 + \sum_{n=1}^{\infty} \lambda^{-n} a^n\right).$$

In this note we generalize an observation of Hochwald and Morell [**4**], which was in turn stimulated by a question of Allan [**1**]:

THEOREM 1. *If A is a centre-commutative ring, if $D \subseteq A$ is a semigroup for which $A^{-1} \subseteq D \subseteq \overline{A}$, and if $g \colon D \to A$ is a mapping which satisfies, for each $a \in A$,*

$$(1.1)\qquad a = ag(a)a$$

and

$$(1.2)\qquad b \in A^{-1} \cap \operatorname{comm}_A(a) \Rightarrow g(b)g(a) = g(a)g(b),$$

Received by the editors July 13, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46H05, 39B70; Secondary 47D05.

Key words and phrases. Generalized inverse, Banach algebra.

then

$$a \in D \Rightarrow a \in aA^{-1}a. \tag{1.3}$$

PROOF. We claim that for each $a \in D$

$$ag(a) = g(a)a: \tag{1.4}$$

this is because if $\lambda \in \text{Centre}(A)$ with $a - \lambda \in A^{-1}$, then $(a-\lambda)^{-1}$ is invertible and commutes with a, so that

$$g((a-\lambda)^{-1}) = a - \lambda \tag{1.5}$$

and

$$g((a-\lambda)^{-1})g(a) = g(a)g((a-\lambda)^{-1}). \tag{1.6}$$

From (1.5) and (1.6) we get (1.4), which now gives (1.3), since

$$h(a) = g(a)ag(a) + (1 - ag(a)) \tag{1.7}$$

is evidently an invertible generalized inverse for a. □

The proof of Theorem 1 is exactly the argument of Hochwald and Morrel [**4**], who are assuming that A is a Banach algebra and that $g(a)$ is a left inverse for each $a \in D$: their conclusion is that a is invertible. Allan [**1**] assumes that $D = A^{-1}_{\text{left}}$ and concludes that g does not exist: his argument uses complex function theory and is confined to complex algebras. In [**2**] and [**3**] we called an element satisfying the condition (1.3) *decomposably regular*: it is rather easy to see that if a decomposably regular element is either left or right invertible then it must be invertible.

REFERENCES

1. G. R. Allan, *Holomorphic left inverse functions*, Proc. Conf. Banach Algebras and Several Complex Variables, Contemp. Math. vol. 32, Amer. Math. Soc., Providence, R. I., 1984.
2. R. E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Royal Irish Acad. **85A** (1986), 151–176.
3. R. E. Harte, *Regular boundary elements*, Proc. Amer. Math. Soc. **99** (1987), 328–330.
4. S. H. Hochwald and B. B. Morrel, *Some consequences of left invertibility*, Proc. Amer. Math. Soc. **100** (1987), 109–110.

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, CORK, IRELAND