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INVERSES OF GENERATORS

RALPH DELAUBENFELS

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ABSTRACT. Let A be a (possibly unbounded) linear operator on a Banach space X that generates a bounded holomorphic semigroup of angle θ ($0 < \theta \leq \pi/2$).

We show that, if the range of A is dense, then A is one-to-one, and A^{-1} (defined on the range of A) generates a bounded holomorphic semigroup of angle θ, given by

$$e^{zA^{-1}} = \int e^{-w}(wA + z)^{-1} \frac{dw}{2\pi i},$$

over an appropriate curve.

When X is reflexive, it is sufficient that A be one-to-one.

Introduction. A special class of semigroups of operators that frequently arises in applications is holomorphic semigroups. Let $S_\theta \equiv \{re^{i\varphi} | 0 < r < \infty, \ |\varphi| < \theta\}$. The C_0 (strongly continuous) semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ is a *bounded holomorphic semigroup* (*BHS*) *of angle* θ ($0 < \theta \leq \pi/2$) if it extends to a semigroup holomorphic on S_θ, and is strongly continuous and bounded on the closure of S_ψ, whenever $\psi < \theta$.

It is well known that a linear operator may generate a C_0 semigroup of contractions and have a bounded inverse that does not (see Example 2, at the end of this paper). In this paper, we show that, if the (possibly unbounded) linear operator A on a Banach space, generates a BHS of angle θ ($0 < \theta \leq \pi/2$) and has dense range, then it is also one-to-one, and the (possibly unbounded) densely defined inverse also generates a BHS of angle θ. We construct the semigroup generated by A^{-1} explicitly.

When one uses the well-known one-to-one correspondence between generators of C_0 semigroups and well-posed Cauchy problems, one obtains the following. For the class of operators considered in our paper, not only is $du/dt = A(u(t))$ well posed, but $A(du/dt) = u(t)$ is also.

Constructions similar to those in this paper appear in [**1**] and in most proofs of the standard generation theorem for bounded holomorphic semigroups.

We remark that our results are related to the asymptotic behavior of semigroups, since a BHS e^{zA} is stable (that is, $\lim_{t\to\infty} e^{tA}x = 0$, for all x in the domain of A) if and only if the range of A is dense (see [**3**, Chapter A-IV, Corollary 1.14, and the comments preceding it]).

All operators are linear, on a Banach space X. When A is one-to-one, A^{-1} will have domain equal to the range of A. We will write e^{tA} for the semigroup generated

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by A. Basic material on C_0 semigroups may be found in Goldstein [**2**], Pazy [**4**], or van Casteren [**5**].

THEOREM. *Suppose A generates a BHS $\{e^{zA}\}$ of angle θ $(0 < \theta \leq \pi/2)$. If the range of A is dense, then A is one-to-one, and A^{-1} generates a BHS $T(z)$ of angle θ, given by*

$$T(z) = \int_{\Gamma_{\varphi,r}} e^{-w} A(wA + z)^{-1} \frac{dw}{2\pi i}$$

where $r > 0$, $|\arg(z)| + \pi/2 - \theta < \varphi < \pi/2$, and $\Gamma_{\varphi,r}$ is drawn in Figure 1. The integral converges in the operator norm topology. When X is reflexive, then it is sufficient that A be one-to-one.

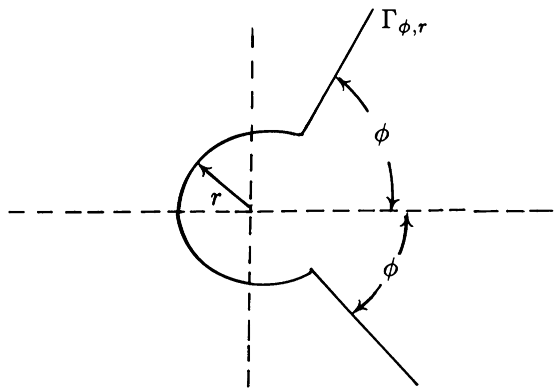

FIGURE 1

PROOF. Suppose the range of A is dense. Since A generates a bounded C_0 semigroup, the closure of the range of A is disjoint from the null-space of A [**5**, Lemma 1.5]. Thus A is one-to-one, so that A^{-1} exists, and is densely defined.

We need to verify that the integral defining $T(z)$ makes sense and converges appropriately. Since A generates a BHS of angle θ, the spectrum of $(-A)$ is contained in $\{z \in C \mid |\arg(z)| \leq \pi/2 - \theta\}$. Thus $(wA + z)^{-1} = w^{-1}(A + z/w)^{-1}$ exists, when $|\arg(z/w)| > \pi/2 - \theta$. On $\Gamma_{\varphi,r}$, $|\arg(w)| \geq \varphi > |\arg(z)| + \pi/2 - \theta$, so that

$$|\arg(z/w)| = |\arg(w)| - |\arg(z)| > \pi/2 - \theta,$$

as desired. Similarly, because A generates a BHS of angle θ, and $\varphi > |\arg(z)| + \pi/2 - \theta$, $\{\|(z/w)(A + z/w)^{-1}\| \mid w \in \Gamma_{\varphi,r}\}$ is bounded. Since

$$A(wA + z)^{-1} = w^{-1}\left(I - \frac{z}{w}\left(A + \frac{z}{w}\right)^{-1}\right),$$

the integrand is continuous, in the operator norm topology, and decays exponentially, as $|w|$ goes to infinity, so that the integral converges in the operator norm.

Note also that, since $(wA+z)^{-1}$ is a holomorphic function of w, the residue theorem implies that $T(z)$ is independent of r and φ, provided they meet the given conditions.

To show that A^{-1} generates a BHS of angle θ, it is sufficient to show that $e^{i\psi}A^{-1}$ generates a bounded C_0 semigroup whenever ψ is real, and $|\psi| < \theta$. To show that $e^{zA^{-1}}$, the semigroup generated by A^{-1}, equals $T(z)$, it is sufficient to show that the semigroup generated by $e^{i\psi}A^{-1}$ is $\{T(te^{i\psi})\}_{t\geq 0}$, when $|\psi| < \theta$.

So fix $\psi \in \mathbf{R}$, $|\psi| < \theta$, and let $S(t) \equiv T(te^{i\psi})$, for $t \geq 0$, with $\pi/2 > \varphi > \pi/2 - \theta + \psi$. We will show the following, for x in the domain of A, $t > 0$.

(1) $\{||S(t)|| \;|t > 0\}$ is bounded.

(2) $S(t)x$ is in the domain of A, and $AS(t)x = S(t)Ax$.

(3) $S(t)Ax$ is a differentiable function of t, with $e^{i\psi}S(t)x = (d/dt)S(t)Ax$.

(4) $\lim_{t\to 0^+} ||S(t)Ax - Ax|| = 0$.

Since A generates a BHS of angle θ, and $\varphi - \psi > \pi/2 - \theta$, there exists a constant M such that

$$(*) \qquad ||(A+z)^{-1}|| \leq M/|z|, \quad \text{whenever } |\arg(z)| \geq \varphi - \psi.$$

The following calculation gives (1):

$$\begin{aligned}
2\pi||S(t)|| &\leq \int_{\Gamma_{\varphi,1}} |e^{-w}|\; ||A(wA + te^{i\psi})^{-1}||\, d|w| \\
&= \int_{\Gamma_{\varphi,1}} |e^{-w}| \left\| A\left(A + \frac{te^{i\psi}}{w}\right)^{-1} \right\| \frac{d|w|}{|w|} \\
&= \int_{\Gamma_{\varphi,1}} |e^{-w}| \left\| I - \frac{te^{i\psi}}{w}\left(A + \frac{te^{i\psi}}{w}\right)^{-1} \right\| \frac{d|w|}{|w|} \\
&\leq \int_{\Gamma_{\varphi,1}} |e^{-w}|(1+M)\frac{d|w|}{|w|} \quad \text{by } (*), \text{ for any positive } t.
\end{aligned}$$

For (2), note that, for any w in $\Gamma_{\varphi,1}$, $(wA + te^{i\psi})^{-1}(Ax)$ is in the domain of A. Since A is closed, and the appropriate integrals converge, we have $S(t)x$ in the domain of A, with

$$\begin{aligned}
AS(t)x &\equiv A\int_{\Gamma_{\varphi,1}} e^{-w}(wA+z)^{-1}Ax\,\frac{dw}{2\pi i} \\
&= \int_{\Gamma_{\varphi,1}} e^{-w}A(wA+z)^{-1}Ax\,\frac{dw}{2\pi i} \\
&= S(t)Ax.
\end{aligned}$$

To establish (3) and (4), it will be convenient to rewrite $S(t)$:

$$\begin{aligned}
\int_{\Gamma_{\varphi,r}} e^{-w}A(wA+te^{i\psi})^{-1}\,dw &= \int_{\Gamma_{\varphi,r}} e^{-w}A\left(\frac{w}{t}A + e^{i\psi}\right)^{-1}\frac{dw}{t} \\
&= \int_{\Gamma_{\varphi,r/t}} e^{-ty}(yA + e^{i\psi})^{-1}\,dy.
\end{aligned}$$

Thus, by Cauchy's theorem, we have

$$S(t) = \int_{\Gamma_{\varphi,r}} e^{-ty} A(yA + e^{i\psi})^{-1} \frac{dy}{2\pi i}, \tag{$**$}$$

for $t, r > 0$, $\psi + (\pi/2 - \theta) < \varphi < \pi/2$.

For (3), note that, since the integrand is a continuously differentiable function of t that decays exponentially as $|y| \to \infty$, we may differentiate $S(t)Ax$ by differentiating inside the integral $(**)$, as follows:

$$\begin{aligned}
2\pi i \frac{d}{dt} S(t)Ax &\equiv \frac{d}{dt} \int_{\Gamma_{\varphi,1}} e^{-ty} A^2 (yA + e^{i\psi})^{-1} x\, dy \\
&= - \int_{\Gamma_{\varphi,1}} e^{-ty} y A^2 (yA + e^{i\psi})^{-1} x\, dy \\
&= - \int_{\Gamma_{\varphi,1}} e^{-ty} (yA + e^{i\psi} - e^{i\psi}) A (yA + e^{i\psi})^{-1} x\, dy \\
&= - \int_{\Gamma_{\varphi,1}} e^{-ty} Ax\, dy + e^{i\psi} \int_{\Gamma_{\varphi,1}} e^{-ty} A(yA + e^{i\psi})^{-1} x\, dy \\
&= 2\pi i e^{i\psi} S(t)x,
\end{aligned}$$

by the residue theorem.

The following calculation gives (4):

$$2\pi i (S(t)Ax - Ax) = \int_{\Gamma_{\varphi,1}} \left(e^{-ty} A(yA - e^{i\psi})^{-1} Ax - \frac{e^{-ty}}{y} Ax \right) dy,$$

by the residue theorem,

$$\begin{aligned}
&= \int_{\Gamma_{\varphi,1}} e^{-ty} [(yA - e^{i\psi} + e^{i\psi})(yA - e^{i\psi})^{-1} Ax - Ax] \frac{dy}{y} \\
&= \int_{\Gamma_{\varphi,1}} e^{-ty} e^{i\psi} (yA - e^{i\psi})^{-1} Ax \frac{dy}{y} \\
&= \int_{\Gamma_{\varphi,1}} e^{-ty} e^{i\psi} [yA(yA - e^{i\psi})^{-1} x] \frac{dy}{y^2}.
\end{aligned}$$

By $(*)$, as argued in the proof of (1), $\|yA(yA - e^{i\psi})^{-1}\|$ is bounded on $\Gamma_{\varphi,1}$. Thus, by dominated convergence,

$$\lim_{t \to 0^+} 2\pi \|S(t)Ax - Ax\| = \left\| \int_{\Gamma_{\varphi,1}} yA(yA - e^{i\psi})^{-1} \frac{dy}{y^2} \right\|.$$

The following calculus of residues argument shows that the integral equals zero. For $N > 1$, let

$$\begin{aligned}
\gamma_N &\equiv \{z \mid |z| = N,\ \varphi \le \arg(z) \le 2\pi - \varphi\}, \\
\theta_N &\equiv \gamma_N \cup \{z \in \Gamma_{\varphi,1} \mid |z| \le N\}.
\end{aligned}$$

Since

$$zA(zA - e^{i\psi})^{-1} = A\left(A - \frac{e^{i\psi}}{z}\right)^{-1} = I + \frac{e^{i\psi}}{z}\left(A - \frac{e^{i\psi}}{z}\right)^{-1}$$

is bounded and holomorphic to the left of $\Gamma_{\varphi,1}$,

$$\int_{\theta_N} yA(yA - e^{i\psi})^{-1} \frac{dy}{y^2} = 0, \quad \text{for all } N,$$

and

$$\lim_{N\to\infty}\int_{\gamma_N} yA(yA - e^{i\psi})^{-1}\,\frac{dy}{y^2} = 0.$$

Thus,

$$\int_{\Gamma_{\varphi,1}} yA(yA - e^{i\psi})^{-1}\,\frac{dy}{y^2} = \lim_{N\to\infty}\int_{(\theta_N-\gamma_N)} yA(yA - e^{i\psi})^{-1}\,\frac{dy}{y^2} = 0.$$

This establishes (4).

This concludes the proof of assertions (1)–(4). Assertion (3) now implies that, for all y in the domain of A^{-1} (equal to the range of A), $(d/dt)S(t)y$ exists and equals $e^{i\psi}S(t)A^{-1}y$, which, by (2), equals $(e^{i\psi}A^{-1})S(t)y$. This implies that $S(t)$ is a semigroup on the domain of A^{-1}, which is dense, so that, by (1), $S(t)$ is a semigroup on X. Similarly, assertions (4) and (1) imply that $S(t)$ is strongly continuous.

Suppose now that y is in the domain of A^{-1}. For t positive, we have

$$\begin{aligned}\frac{1}{t}(S(t)y - y) &= \frac{1}{t}\int_0^t \frac{d}{dr}S(r)y\,dr\\ &= \frac{1}{t}\int_0^t S(r)(e^{i\psi}A^{-1}y)\,dr,\end{aligned}$$

so that, by the continuity of $S(t)$,

$$\lim_{t\to 0^+}\frac{1}{t}(S(t)y - y) = e^{i\psi}A^{-1}y.$$

This implies that an extension of $(e^{i\psi}A^{-1})$ generates $S(t)$. Thus $(e^{i\psi}A^{-1})$ is dissipative, with respect to an equivalent norm. Note now that the range of $(I - e^{i\psi}A^{-1})$ equals $\{(I - e^{i\psi}A^{-1})(Ax) \mid x$ is in the domain of $A\}$, which equals the range of $(A - e^{i\psi}I)$, which is X, since $|\psi| < \pi/2$, and A generates a bounded C_0 semigroup. Since $e^{i\psi}A^{-1}$ is densely defined, the Lumer-Phillips theorem implies that $(e^{i\psi}A^{-1})$ generates a bounded C_0 semigroup. This semigroup is $S(t)$, since we have already shown that an extension of $(e^{i\psi}A^{-1})$ generates $S(t)$.

By the comments near the beginning of the proof, this shows that A^{-1} generates a BHS of angle θ, given by $T(z)$.

Now suppose X is reflexive, and A is one-to-one. By [**2**, Theorem 8.20, p. 58], the range of A is dense.

EXAMPLE 1. Let A be the Laplacian Δ on $L^p(\mathbf{R}^n)$ $(1 \le p < \infty)$. It is well known that A generates a BHS of angle $\pi/2$.

If $1 < p < \infty$, then, since A is one-to-one, and $L^p(\mathbf{R}^n)$ is reflexive, our theorem implies that A^{-1} generates a BHS of angle $\pi/2$.

When $p = 1$, since A^* is not one-to-one, the range of A is not dense, so that A^{-1} is not densely defined, thus cannot generate a C_0 semigroup. Note that this demonstrates that reflexivity is necessary in the last sentence of the theorem.

EXAMPLE 2. This is an example of an operator that generates a C_0 contraction semigroup, with a bounded inverse that does not generate a C_0 contraction semigroup. This shows that the collection of invertible generators of C_0 contraction semigroups, unlike the collection of invertible generators of bounded holomorphic semigroups, is not closed under inversion.

On $C_0(\mathbf{R})$, let $A \equiv (d/dx - I)$, the generator of $(e^{tA}f)(x) \equiv e^{-t}f(x+t)$. We have

$$(A^{-1}f)(x) = -e^x \int_x^\infty e^{-t} f(t)\, dt.$$

Define $\varphi \in C_0(\mathbf{R})^*$ by $\varphi(f) \equiv f(0)$. It is not difficult to see that there exists g in $C_0(\mathbf{R})$ such that $\varphi(g) = \|g\|_\infty$, while $\varphi(A^{-1}g) = -\int_0^\infty e^{-t}g(t)\, dt > 0$. This means that A^{-1} is not dissipative, and thus cannot generate a C_0 contraction semigroup.

Example 2 shows that one possible analogue of our theorem is false. More natural analogues are in the following open questions.

OPEN QUESTIONS. What role does analyticity play in the theorem? That is, suppose A is one-to-one with dense range and generates a bounded C_0 semigroup. Does A^{-1} generate a bounded C_0 semigroup? Does A^{-1} generate a C_0 semigroup?

REMARK. The following special case provides motivation for both the theorem of this paper, and the open questions above.

Suppose A is a normal operator on a Hilbert space. By the spectral theorem, we may assume that there exists a measure space (Ω, μ), and a complex-valued measurable function f so that A is multiplication by f, that is, $Ag = fg$, with A having maximal domain. The operator A generates a BHS of angle θ if and only if the essential range of f is contained in $-\overline{S}_{(\pi/2-\theta)} = \{-w \mid |\arg(w)| \leq \pi/2 - \theta\}$, since e^{zA} is multiplication by e^{zf}. A is one-to-one if and only if f is nonzero a.e. Since A^{-1} is then multiplication by $1/f$, and, for any x in Ω, $f(x)$ is in $-\overline{S}_{\pi/2-\theta}$ if and only if $1/f(x)$ is in $-\overline{S}_{\pi/2-\theta}$, it follows that A generates a BHS of angle θ if and only if A^{-1} generates a BHS of angle θ.

For the same operator A, A generates a bounded C_0 semigroup if and only if the essential range of f is contained in the closed left half-plane ($\overline{\text{LHP}}$). Since $1/f(x)$ is in $\overline{\text{LHP}}$ if and only if $f(x)$ is in $\overline{\text{LHP}}$, we also conclude that A generates a bounded C_0 semigroup if and only if A^{-1} generates a bounded C_0 semigroup.

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DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701