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COMPARISON THEOREMS FOR CAUSAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Weak and strong comparison theorems are proved for causal functional differential equations

Introduction. In [**1**], comparison theorems were derived for initial value problems associated with ordinary differential equations. In this note, the ideas and techniques are applied to causal functional differential equations of the form

$$\frac{d}{dt}y(t) = f(t,y), \qquad t \in (a,b),$$

where $y \in S_n$ the space of continuous maps from $[a,b] \subset R$ into R^n, which are differentiable on (a,b), and f is a map from $[a,b] \times S_n$ into R^n, which is causal in the sense that

$$f(t,u) = f(t,v), \qquad t \in [a,b],$$

for all $u, v \in S_n$ such that $u(s) = v(s)$ when $a \leq s \leq t$.

If $y, z \in S_n$ we may define the functions $\overline{f}(t,y,z)$, $\underline{f}(t,y,z)$ mapping $[a,b] \times S_n^2$ into R^n as follows; componentwise,

$$\overline{f}(t,y,z) = \sup f(t,\theta), \quad \underline{f}(t,y,z) = \inf f(t,\theta),$$

where $\theta \in S_n$ and for every component i, and each point s in $[a,b]$,

$$\min(y_i(s), z_i(s)) \leq \theta_i(s) \leq \max(y_i(s), z_i(s)).$$

1.

WEAK COMPARISON THEOREM *Suppose $\underline{y}, y, \overline{y} \in S_n$, $f(t,y)$ is causal and*

$$\underline{y}(a) < y(a) < \overline{y}(a), \tag{1}$$

$$\frac{d}{dt}\underline{y}(t) - \underline{f}(t,\underline{y},\overline{y}) < \frac{d}{dt}y(t) - f(t,y) < \frac{d}{dt}\overline{y}(t) - \overline{f}(t,\underline{y},\overline{y}) \quad \textit{on } (a,b]. \tag{2}$$

Then

$$\underline{y}(t) < y(t) < \overline{y}(t) \quad \textit{on } [a,b]. \tag{3}$$

PROOF. If the inequality (3) does not hold, there is a point x in $(a,b]$ such that (3) holds in $[a,x)$, but $y_i(x) = \overline{y}_i(x)$ or $\underline{y}_i(x)$ for some component y_i of y. Suppose $y_i(x) = \overline{y}_i(x)$.

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Then

$$\frac{d}{dt}y_i(x) \geq \frac{d}{dt}\overline{y}_i(x),$$

and hence

$$\frac{d}{dt}y_i(x) - f_i(x,y) \geq \frac{d}{dt}\overline{y}_i(x) - \overline{f}_i(x,\underline{y},\overline{y}),$$

since $\overline{f}_i(x,\underline{y},\overline{y}) = \sup f_i(x,\theta)$ for $\underline{y} \leq \theta \leq \overline{y}$ and y is a candidate in the search for the supremum since f is causal. This violates assumption (2) and a similar argument makes the alternative assumption $y_i(x) = \underline{y}_i(x)$ likewise untenable.

2.

STRONG COMPARISON THEOREM *The weak theorem can be made stronger if $f(t,u)$ satisfies a Lipschitz condition of the form*

$$\begin{aligned} ||f(t,u) - f(t,v)|| &= \sup_i |f_i(t,u) - f_i(t,v)| \leq K||u-v||_t \\ &= K\sup\{|u_j(s) - v_j(s)| \ |1 \leq j \leq n, s \in (a,t)\}. \end{aligned}$$

Suppose $\underline{y}, \overline{y}, y \in S_n$; $f(t,y)$ is a causal locally Lipschitz continuous map in the sense described above, and

(1) $$\underline{y}(a) < y(a) < \overline{y}(a),$$

(2) $$\frac{d}{dt}\underline{y}(t) - \underline{f}(t,\underline{y},\overline{y}) \leq \frac{d}{dt}y - f(t,y) \leq \frac{d}{dt}\overline{y} - \overline{f}(t,\underline{y},\overline{y}).$$

Then

(3) $$\underline{y} < y < \overline{y} \quad \textit{on } [a,b].$$

PROOF. Let $w \in R^n$ be such that $w > 0$ and $2w < \overline{y}(a) - y(a)$ and $y(a) - \underline{y}(a)$ componentwise. Define $\overline{u} = \overline{y} - e^{-2K(t-a)}w$, $\underline{u} = \underline{y} + e^{-2K(t-a)}w$, and note that $\underline{y}(a) < \overline{u}(a) < y(a) < \overline{u}(a) < \overline{y}(a)$.

Since $\overline{u}$, $\underline{u}$ are continuous on $[a,b]$ and $\overline{u}(a) > \underline{u}(a)$, there is an x in $[a,b]$ such that $[a,x]$ is the maximal interval for which $\underline{u}(s) \leq y(s) \leq \overline{u}(s)$ for all s in $[a,x]$. Consider the expression $\overline{E}$ defined by

$$\begin{aligned} \overline{E} &= \frac{d}{dt}\overline{u} - \overline{f}(t,\underline{u},\overline{u}) \\ &= \left\{\frac{d}{dt}\overline{y} - \overline{f}(t,\underline{y},\overline{y})\right\} + 2Ke^{-2K(t-a)}w + \left\{\overline{f}(t,\underline{y},\overline{y}) - \overline{f}(t,\underline{u},\overline{u})\right\} \\ &\geq \left\{\frac{d}{dt}y - f(t,y)\right\} + 2Ke^{-2K(t-a)}w + \left\{\overline{f}(t,\underline{y},\overline{y}) - \overline{f}(t,\underline{u},\overline{u})\right\}. \end{aligned}$$

By definition,

$$\begin{aligned} &\overline{f}_i(t,\underline{y},\overline{y}) = \sup f_i(t,\theta) \\ &\qquad \text{for } \{\theta\colon \theta \in S_n \text{ and } \min(\underline{y}(s),\overline{y}(s)) \leq \theta(s) \leq \max(\underline{y}(s),\overline{y}(s))\}; \\ &\overline{f}_i(t,\underline{u},\overline{u}) = \sup f_i(t,\theta) \\ &\qquad \text{for } \{\theta\colon \theta \in S_n \text{ and } \min(\underline{u}(s),\overline{u}(s)) \leq \theta(s) \leq \max(\underline{u}(s),\overline{u}(s))\}. \end{aligned}$$

On the closed interval $[a, x]$, $\overline{f}_i(t, \underline{u}, \overline{u}) \leq \overline{f}_i(t, \underline{y}, \overline{y})$, and hence

$$\overline{E} > \frac{d}{dt}y - f(t, y) \quad \text{on } [a, x].$$

In similar fashion

$$\underline{E} = \frac{d}{dt}\underline{u} - \underline{f}(t, \underline{u}, \overline{u}) < \frac{d}{dt}y - f(t, y) \quad \text{on } [a, x].$$

By the weak comparison theorem $\underline{u} < y < \overline{u}$ on $[a, x]$ and since $[a, x]$ is maximal, we must conclude $x = b$, and $\underline{y} < \underline{u} < y < \overline{u} < \overline{y}$ on $[a, b]$.

3. An example. Consider the differential delay equation

$$\begin{aligned} \frac{d}{dt}y(t) &= A(t)y(t-h), \qquad t \in (h, b),\ 0 < h < b, \\ &= A(t)g(t), \qquad t \in (0, h],\ y(0) = g(h), \end{aligned} \tag{3.1}$$

where $y \in S_n$, and $A(t)$ is an $n \times n$ matrix. For this example,

$$f(t, y) \equiv A(t)y(t-h) \quad \text{when } t \in (h, b)$$

and

$$\begin{aligned} \overline{f}_i(t, \underline{y}, \overline{y}) &\equiv \sup_{\underline{y} \leq \theta \leq \overline{y}} \sum_{j=1}^{n} A(t)_{ij}\theta_j(t-h) \\ &= [A^+(t)\overline{y}(t-h) + A^-(t)\underline{y}(t-h)]_i, \end{aligned}$$

where $A^+(t)$ is the matrix with $A^+(t)_{ij} = A(t)_{ij}$ if $A(t)_{ij} \geq 0$ and $A^+(t)_{ij} = 0$ otherwise. $A^-(t)$ similarly is the matrix of negative elements of $A(t)$ and evidently $A(t) \equiv A^+(t) + A^-(t)$. Likewise,

$$\underline{f}(t, \underline{y}, \overline{y}) = A^+(t)\underline{y}(t-h) + A^-(t)\overline{y}(t-h).$$

We are led to associate the $2n$ order system

$$\begin{aligned} &\frac{d}{dt}\overline{y} \geq A^+(t)\overline{y}(t-h) + A^-(t)\underline{y}(t-h), \qquad t \in (h, b), \\ &\frac{d}{dt}\underline{y} \leq A^+(t)\underline{y}(t-h) + A^-(t)\overline{y}(t-h), \\ &\frac{d}{dt}\overline{y} \geq A^+(t)\overline{g}(t) + A^-(t)\underline{g}(t), \qquad t \in (0, h], \\ &\frac{d}{dt}\underline{y} \leq A^+(t)\underline{g}(t) + A^-(t)\overline{g}(t), \qquad \overline{g}(s) \geq g(s) \geq \underline{g}(s), \quad s \in (0, h], \\ &\overline{y}(0) \geq \overline{g}(h), \qquad \underline{y}(0) \leq \underline{g}(h), \end{aligned} \tag{3.2}$$

with system (3.1).

If $z(t) = \overline{y}(t) - \underline{y}(t)$, we see that

$$\begin{aligned} &\frac{d}{dt}z \geq (A^+(t) - A^-(t))z(t-h) \quad \text{for } t \in (h, b), \\ &\frac{d}{dt}z \geq (A^+(t) - A^-(t))(\overline{g}(t) - \underline{g}(t)) \quad \text{for } t \in (0, h] \\ &z(0) \geq \overline{g}(h) - \underline{g}(h) \geq 0. \end{aligned} \tag{3.3}$$

One half of the strong comparison theorem applied to (3.3) shows that $z \geq 0$ on $(0, b)$ and $(d/dz)z \geq 0$ and $(0, b)$ and so, if

$$\frac{d}{dt}\overline{z}(t) = |A(t)|\overline{z}(t) \tag{3.4}$$

where $|A| \equiv A^+ - A^-$, then

$$\begin{aligned} \frac{d}{dt}\overline{z}(t) &= |A(t)|\overline{z}(t-h) + |A(t)|(\overline{z}(t) - \overline{z}(t-h)) \\ &> |A(t)|\overline{z}(t-h). \end{aligned}$$

Thus the differential equation (3.4) gives rise to upper bounds on $z(t)$.

REFERENCES

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