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SOME EXPLICIT REAL ANALYTIC TRIVIALIZATIONS OF THE TEICHMÜLLER CURVES

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ABSTRACT. The barycentric extension of homeomorphisms from the unit circle to the closed unit disk produces real analytic trivializations of the Teichmüller curves.

1. Introduction. In part because of their universal properties (see [**5**, **6**, and **9**]) the Teichmüller curves have played a central role in the deformation theory of Riemann surfaces. Since Teichmüller spaces are contractible, the Teichmüller curves are topologically trivial fibrations. At times (see for example [**12**]), smooth trivializations are useful. Our purpose here is to present explicit real analytic trivializations. These trivializations, which we obtain from the barycentric extension introduced by Douady and the author in [**3**], have invariance properties that may be useful. We present our basic construction in the next section. The final section describes the resulting trivializations.

We assume some familiarity with the Ahlfors-Bers theory of Teichmüller spaces. Facts for which no reference is given can be found in the recent books [**8**, **10**, or **11**].

2. A real analytic diffeomorphism. Let G be the group of holomorphic automorphisms of the open unit disk Δ in $\mathbf{C}$, and let S^1 be the boundary of Δ. As usual, $C(S^1, \mathbf{C})$ is the Banach space of continuous complex valued functions on S^1, with the sup norm, and $\mathscr{H}(S^1)$ is the group of homeomorphisms of S^1 onto itself. If Γ is a Fuchsian group (discrete subgroup of G), we define the Teichmüller space $T(\Gamma)$ to be the set of φ in $\mathscr{H}(S^1)$ that fix the points 1, i and -1 and have a quasiconformal extension $w\colon \overline{\Delta} \to \overline{\Delta}$ that satisfies

$$w \circ \gamma \circ w^{-1} \in G \quad \text{for all } \gamma \in \Gamma. \tag{2.1}$$

We denote by 1 the trivial subgroup of G; thus $T(1)$ is the set of φ in $\mathscr{H}(S^1)$ that fix 1, i, and -1 and have a quasiconformal extension to $\overline{\Delta}$. The Ahlfors-Bers theory tells us that $T(1)$ is an infinite dimensional complex Banach manifold that contains each space $T(\Gamma)$ as a closed (possibly infinite dimensional) complex submanifold.

For each φ in $\mathscr{H}(S^1)$ let $\operatorname{ex}(\varphi)\colon \overline{\Delta} \to \overline{\Delta}$ be the barycentric extension defined in [**3**]. We recall that for w and z in Δ and φ in $\mathscr{H}(S^1)$, $w = \operatorname{ex}(\varphi)(z)$ if and only if

$$\frac{1}{2\pi} \int_{S^1} \left(\frac{\varphi(\varsigma) - w}{1 - \bar{w}\varphi(\varsigma)} \right) \frac{1 - |z|^2}{|z - \varsigma|^2} \, |d\varsigma| = 0. \tag{2.2}$$

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Our basic result is

THEOREM 1. *The map* $F: T(1) \times \Delta \to T(1) \times \Delta$ *defined by*

$$F(\varphi, z) = (\varphi, \mathrm{ex}(\varphi)(z)) \tag{2.3}$$

is a real analytic diffeomorphism.

PROOF. To prove the analyticity we must first describe the complex structure of $T(1)$. Let $M(1)$ be the open unit ball in the complex Banach space $L^\infty(\Delta, \mathbf{C})$. For each function μ in $M(1)$ there is a unique quasiconformal map f^μ of $\overline{\Delta}$ onto itself that fixes the points 1, i, and -1 and satisfies the Beltrami equation $f_{\bar{z}} = \mu f_z$ in Δ. Let φ^μ be the restriction $f^\mu|S^1$, and define the surjective map $\Phi: M(1) \to T(1)$ by $\Phi(\mu) = \varphi^\mu$. The complex structure of $T(1)$ is uniquely characterized by the statement that Φ is a holomorphic map with local holomorphic right inverses.

Consider the function f on $M(1) \times \Delta$ defined by $f(\mu, z) = \mathrm{ex}(\varphi^\mu)(z)$. The map $(\mu, z) \mapsto (\varphi^\mu, z)$ from $M(1) \times \Delta$ to $C(S^1, \mathbf{C}) \times \Delta$ is real analytic, as a consequence of Lemma 18 and Theorem 11 of [**1**]. In addition, the equation (2.2) satisfied by $w = \mathrm{ex}(\varphi)(z)$ implies (see p. 31 of [**3**] for a proof) that for any point (φ_0, z_0) in $\mathscr{H}(S^1) \times \Delta$ there is a real analytic function $h(\varphi, z)$ defined in a neighborhood W of (φ_0, z_0) in $C(S^1, \mathbf{C}) \times \Delta$ and satisfying $h(\varphi, z) = \mathrm{ex}(\varphi)(z)$ if $(\varphi, z) \in W$ and $\varphi \in \mathscr{H}(S^1)$. These two observations imply that f is a real analytic function. Consequently, the function $(\varphi, z) \mapsto \mathrm{ex}(\varphi)(z)$ on $T(1) \times \Delta$ and the map F defined by (2.3) are real analytic.

Finally, F is a diffeomorphism because $\mathrm{ex}(\varphi)$ is a diffeomorphism of Δ onto itself for each fixed φ, by Theorem 1 of [**3**]. Q.E.D.

REMARK. The complex structure of $T(1)$ is *not* induced by the inclusion of $T(1)$ in $C(S^1, \mathbf{C})$. That inclusion map is not holomorphic, nor is the evaluation function $\varphi \mapsto \varphi(z)$, for any fixed z in $S^1 \backslash \{1, i, -1\}$.

3. The Teichmüller curves. Each φ in $T(1)$ uniquely determines a quasiconformal homeomorphism w_φ of the Riemann sphere S^2 onto itself that fixes the points 1, i, and -1 and satisfies the following two conditions:

$$w_\varphi \text{ is conformal in the exterior of } \Delta, \tag{3.1}$$

$$h_\varphi = w_\varphi \circ \mathrm{ex}(\varphi)^{-1} \text{ is conformal in } \Delta. \tag{3.2}$$

The Bers fiber space of any Fuchsian group Γ is the open complex submanifold $F(\Gamma)$ of $T(\Gamma) \times S^2$ defined by

$$F(\Gamma) = \{(\varphi, \varsigma); \varphi \in T(\Gamma) \text{ and } \varsigma \in w_\varphi(\Delta)\}.$$

According to Bers [**2**], the group Γ acts on $F(\Gamma)$ as a group of biholomorphic mappings in the following way. Condition (2.1) implies that for each γ in Γ and φ in $T(\Gamma)$ there is a unique automorphism γ_φ of Δ such that

$$\gamma_\varphi \circ \varphi = \varphi \circ \gamma \quad \text{on } S^1.$$

The conformal naturality of the barycentric extension (see [**3**]) therefore implies that

$$\gamma_\varphi \circ \mathrm{ex}(\varphi) = \mathrm{ex}(\varphi) \circ \gamma \quad \text{in } \Delta.$$

Condition (3.2) now implies that

$$\hat{\gamma}_\varphi = h_\varphi \circ \gamma_\varphi \circ h_\varphi^{-1} = w_\varphi \circ \gamma \circ w_\varphi^{-1} \tag{3.3}$$

is a holomorphic automorphism of $h_\varphi(\Delta) = w_\varphi(\Delta)$. The Bers action of Γ on $F(\Gamma)$ is given by the formula

$$\gamma \cdot (\varphi, \varsigma) = (\varphi, \hat{\gamma}_\varphi(\varsigma)) \quad \text{for all } \gamma \in \Gamma \text{ and } (\varphi, \varsigma) \in F(\Gamma). \tag{3.4}$$

The action of Γ on $F(\Gamma)$ is properly discontinuous, and the quotient space $V(\Gamma) = F(\Gamma)/\Gamma$ is a complex Banach manifold. The projection $(\varphi, \varsigma) \mapsto \varphi$ of $F(\Gamma)$ onto $T(\Gamma)$ induces a well defined holomorphic surjection $\pi\colon V(\Gamma) \to T(\Gamma)$, which defines a holomorphic family of Riemann surfaces (see §1.2 of **[5]** for a discussion of holomorphic families). That family $\pi\colon V(\Gamma) \to T(\Gamma)$ is the Teichmüller curve over $T(\Gamma)$.

REMARK. Since $T(\Gamma)$ is a closed complex submanifold of $T(1)$, the Bers fiber space $F(\Gamma)$ is a closed complex submanifold of $F(1)$ for any Fuchsian group Γ.

4. The trivializations. Fix a Fuchsian group Γ, and let X be the quotient Riemann surface Δ/Γ. A trivialization of the Teichmüller curve over $T(\Gamma)$ is a homeomorphism $\theta\colon T(\Gamma) \times X \to V(\Gamma)$ such that $\pi(\theta(\varphi, x)) = \varphi$ for all (φ, x) in $T(\Gamma) \times X$. We observe that $T(\Gamma) \times X$ is the quotient of $T(\Gamma) \times \Delta$ by the Γ-action

$$\gamma(\varphi, z) = (\varphi, \gamma z) \quad \text{for all } \gamma \in \Gamma \text{ and } (\varphi, z) \in T(\Gamma) \times \Delta. \tag{4.1}$$

THEOREM 2. *The map $(\varphi, z) \mapsto (\varphi, w_\varphi(z))$ from $T(\Gamma) \times \Delta$ to $F(\Gamma)$ is a Γ-equivariant real analytic diffeomorphism. The induced map $\theta\colon T(\Gamma) \times X \to V(\Gamma)$ is a trivialization of the Teichmüller curve over $T(\Gamma)$. If the group Γ is torsion-free, then θ is a real analytic diffeomorphism.*

PROOF. Formulas (3.3), (3.4), and (4.1) show that the map $(\varphi, z) \mapsto (\varphi, w_\varphi(z))$ is Γ-equivariant. In proving that it is a real analytic diffeomorphism we may assume that $\Gamma = 1$, since $T(\Gamma) \times \Delta$ and its image $F(\Gamma)$ are closed complex submanifolds of $T(1) \times \Delta$ and $F(1)$. Using condition (3.2) we write $(\varphi, w_\varphi(z)) = (H \circ F)(\varphi, z)$, where $F\colon T(1) \times \Delta \to T(1) \times \Delta$ is the map (2.3), $H\colon T(1) \times \Delta \to F(1)$ is the map

$$H(\varphi, z) = (\varphi, h_\varphi(z)),$$

and h_φ is the conformal map of Δ onto $w_\varphi(\Delta)$ that fixes the points 1, i, and -1. Theorem 1 tells us that F is a real analytic diffeomorphism. So is H, by Lemmas 6.3 and 6.4 of Bers **[2]**. (An alternative proof is indicated in §5.3 of **[4]**. In both references the role of Δ is played by the upper half plane U. Conjugation by a Möbius transformation transfers the results from one setting to the other.) Therefore $H \circ F$ is a real analytic diffeomorphism, as required.

The induced homeomorphism $\theta\colon T(\Gamma) \times X \to V(\Gamma)$ is obviously a trivialization of the Teichmüller curve. Finally, if Γ is torsion-free the quotient maps from $T(\Gamma) \times \Delta$ to $T(\Gamma) \times X$ and from $F(\Gamma)$ to $V(\Gamma)$ are locally biholomorphic, so θ is a real analytic diffeomorphism. Q.E.D.

REMARKS. (1) If Γ is the group of order two generated by $z \mapsto -z$, then θ is not a diffeomorphism.

(2) If Γ has torsion and the universal cover of $X = \Delta/\Gamma$ is Δ, the methods of §4 in **[7]** allow us to write the Teichmüller curve $V(\Gamma) \to T(\Gamma)$ as the pullback of a Teichmüller curve $V(\Gamma_1) \to T(\Gamma_1)$ with Γ_1 torsion-free and $X = \Delta/\Gamma_1$. We can then pull back a real analytic trivialization from $V(\Gamma_1)$ to $V(\Gamma)$. Similar methods can be used if the universal cover of X is $\mathbf{C}$ or S^2.

(3) If the groups Γ_1 and Γ_2 are conjugate in G, the natural isomorphisms from $T(\Gamma_1) \times \Delta/\Gamma_1$ to $T(\Gamma_2) \times \Delta/\Gamma_2$ and from $F(\Gamma_1)$ to $F(\Gamma_2)$ commute with our trivializations θ, by conformal naturality (see **[3]**). Similar statements hold if $\Gamma_1 \subset \Gamma_2$.

REFERENCES

1. L. V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404.
2. L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89–126.
3. A. Douady and C. J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.
4. C. J. Earle, *On quasiconformal extensions of the Beurling-Ahlfors type*, Contributions to Analysis, Academic Press, New York and London, 1974, pp. 99–105.
5. C. J. Earle and R. S. Fowler, *Holomorphic families of open Riemann surfaces*, Math. Ann. **270** (1985), 249–273.
6. ——, *A new characterization of infinite dimensional Teichmüller spaces*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **10** (1985), 149–153.
7. C. J. Earle and I. Kra, *On sections of some holomorphic families of closed Riemann surfaces*, Acta Math. **137** (1976), 49–79.
8. F. P. Gardiner, *Teichmüller theory and quadratic differentials*, Wiley-Interscience, New York, 1987.
9. A. Grothendieck, *Techniques de construction en géométrie analytique*, Séminaire H. Cartan, 13ème année: 1960/61.
10. O. Lehto, *Univalent functions and Teichmüller spaces*, Springer-Verlag, Berlin and New York, 1987.
11. S. Nag, *The complex analytic theory of Teichmüller spaces*, Wiley-Interscience, Wiley, New York, 1988.
12. P. L. Sipe, *Roots of the canonical bundle of the universal Teichmüller curve and certain subgroups of the mapping class group*, Math. Ann. **260** (1982), 67–92.

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