PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

JUMP DISTRIBUTIONS ON $[-1,1]$ WHOSE ORTHOGONAL POLYNOMIALS HAVE LEADING COEFFICIENTS WITH GIVEN ASYMPTOTIC BEHAVIOR

D. S. LUBINSKY

(Communicated by Richard R. Goldberg)

ABSTRACT. We construct an explicit even jump distribution $d\alpha(x)$ on $[-1,1]$ for which $\gamma_n(d\alpha)$, the leading coefficient of the nth orthonormal polynomial for $d\alpha(x)$, has any given asymptotic behavior. For example, if $\{\xi_n\}_1^\infty$ is a strictly increasing sequence with limit ∞, and $\xi_{3^{n+1}}/\xi_{3^n} \to 1$ as $n \to \infty$, we can ensure that $\lim_{n\to\infty} \gamma_n(d\alpha)2^{-n}/\xi_n = 1$. As a consequence, we have $\lim_{n\to\infty} \gamma_{n-1}(d\alpha)/\gamma_n(d\alpha) = 1/2$; that is, $d\alpha$ belongs to Nevai's class $\mathscr{M}$. This positively and explicitly answers a question of Al. Magnus.

Statement of results. Let $\alpha(x)$ be a monotone increasing function on $[-1,1]$ with infinitely many points of increase and with all moments of $d\alpha(x)$ finite. Then there exists a sequence of orthonormal polynomials

$$p_n(d\alpha,x) = \gamma_n(d\alpha)x^n + \cdots, \qquad \gamma_n(d\alpha) > 0,\ n = 1,2,3,\ldots,$$

satisfying

$$\int_{-1}^{1} p_n(d\alpha,x)p_m(d\alpha,x)\, d\alpha(x) = \delta_{mn}, \qquad m,n = 0,1,2,\ldots .$$

These polynomials satisfy a three-term recurrence relation

$$xp_n(d\alpha,x) = A_n(d\alpha)p_{n+1}(d\alpha,x) + B_n(d\alpha)p_n(d\alpha,x) + A_{n-1}(d\alpha)p_{n-1}(d\alpha,x), \qquad n = 1,2,3,\ldots,$$

where

$$A_n(d\alpha) := \gamma_{n-1}(d\alpha)/\gamma_n(d\alpha); \qquad B_n(d\alpha) := \int_{-1}^{1} xp_n(d\alpha,x)^2\, d\alpha(x).$$

The class $\mathscr{M} = \mathscr{M}(0,1)$ of $d\alpha(x)$ for which

$$\lim_{n\to\infty} A_n(d\alpha) = 1/2 \quad \text{and} \quad \lim_{n\to\infty} B_n(d\alpha) = 0 \tag{1.1}$$

was introduced by Nevai in his memoir [**8**] and extensively studied there. Subsequent papers, especially by Mate, Nevai and Totik (see the surveys [**4**, **9**]), continued these investigations. Recently, Al. Magnus [**6**], who proved [**5**] that general complex weights may also satisfy (1.1), raised the question as to whether there is a pure jump distribution in Nevai's class. A probabilistic solution to this problem has

Received by the editors July 28, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42C05.
Key words and phrases. Orthogonal polynomials, jump distributions, asymptotics, Nevai's class, leading coefficients, Jacobi matrices, discrete scattering.

been given by Delyon, Simon and Souillard [**1**]. This paper grew out of attempts to solve Magnus' problem explicitly. We note that the problem has connections with spectral theory (especially of Jacobi matrices), with scattering theory, and with operator theory; see [**1**, **7**, **9**] and references therein. J. Von Neumann [**12**] presented a related example about perturbations of compact operators.

For $n = 1,2,3,\dots$, we let $T_n(x) := \cos(n \arccos x)$ denote the nth Chebyshev polynomial and let

$$I_n[f] := \frac{\pi}{n}\sum_{j=1}^{n} f\left(\cos\left(\frac{2j-1}{2n}\pi\right)\right) \tag{1.2}$$

denote the associated Gauss quadrature rule of order n. Let $\{\eta_j\}_1^\infty$ be a sequence of positive numbers with

$$\sum_{j=1}^{\infty} \eta_j = 1. \tag{1.3}$$

We define a mass distribution $d\alpha(x)$ on $[-1,1]$ by

$$\int_{-1}^{1} f(x)\, d\alpha(x) := \pi^{-1}\sum_{j=1}^{\infty} \eta_j I_{3^j}[f], \tag{1.4}$$

for each f continuous in $[-1,1]$. It is clear from (1.4) that $d\alpha(x)$ is a pure even jump distribution with jumps at the zeros of $T_{3^j}(x)$, $j = 1,2,3,\dots$. We shall prove

THEOREM 1.1. *Let $\{\xi_n\}_1^\infty$ be a strictly increasing sequence of positive numbers satisfying*

$$\lim_{n\to\infty} \xi_n = \infty \tag{1.5}$$

and

$$\lim_{n\to\infty} \xi_{3^{n+1}}/\xi_{3^n} = 1. \tag{1.6}$$

Then there is an even jump distribution $d\alpha$ of the form (1.2) *to* (1.4) *satisfying*

$$\lim_{n\to\infty} \gamma_n(d\alpha)/\{2^n \xi_n\} = 1. \tag{1.7}$$

In particular, $d\alpha \in \mathscr{M}$.

One may form a perspective of the above result if one recalls the equivalence of the following two statements, well known in Szegö's theory [**3**, p. 172]:

(I) $\{\gamma_n(d\alpha)/2^n\}_1^\infty$ (or even some subsequence) is bounded.

(II) $\int_{-1}^{1} \log \alpha'(x)(1-x^2)^{-1/2}\, dx > -\infty$.

In particular, for pure jump distributions, one must have

$$\lim_{n\to\infty} \gamma_n(d\alpha)/2^n = \infty.$$

Theorem 1.1 shows that there are pure jump distributions for which $\gamma_n(d\alpha)/2^n$ nevertheless grows smoothly and arbitrarily slowly. Of course, $\gamma_n(d\alpha)$ may also grow arbitrarily fast:

THEOREM 1.2. *Let* $\{\xi_n\}_1^\infty$ *be a strictly increasing sequence of positive numbers satisfying* (1.5). *Then there is an even jump distribution* $d\alpha$ *of the form* (1.2) *to* (1.4) *satisfying*

$$1 \le \gamma_n(d\alpha)/\{2^n \xi_{\langle \log_3 n \rangle}\} \le \sqrt{3}, \qquad n = 3, 4, 5, \dots . \tag{1.8}$$

In the above result, as in the sequel, $\langle x \rangle$ denotes the largest integer $\le x$. Both Theorems 1.1 and 1.2 are corollaries of the following.

THEOREM 1.3. *Let* $d\alpha(x)$ *be defined by* (1.2) *to* (1.4). *Then:*

(a) *For* $n = 3, 4, 5, \dots$, *we have*

$$\frac{1}{\sqrt{3}} \le \gamma_n(d\alpha) 2^{-n+1/2} \left\{ \sum_{j=\langle \log_3 n \rangle + 1}^{\infty} \eta_j \right\}^{1/2} \le 1. \tag{1.9}$$

When n *is a positive integer power of* 3, *the second inequality in* (1.9) *becomes an equality, and* $p_n(d\alpha, x)$ *has the same zeros as* $T_n(x)$.

(b) *If also*

$$\lim_{n\to\infty} \eta_n \Big/ \sum_{j=n+1}^{\infty} \eta_j = 0, \tag{1.10}$$

then

$$\lim_{n\to\infty} \gamma_n(d\alpha) 2^{-n+1/2} \left\{ \sum_{j=\langle \log_3 n \rangle + 1}^{\infty} \eta_j \right\}^{1/2} = 1. \tag{1.11}$$

One feature of Theorem 1.3(a) is that $\{p_{3^n}(d\alpha, x)\}_1^\infty$ has "arcsin distribution", that is, their zeros behave asymptotically like (and in this case are) the zeros of the Chebyshev polynomials. Such behaviour of the zeros is typically associated with the limit relation

$$\lim_{n\to\infty} \gamma_{3^n}(d\alpha)^{1/3^n} = 2,$$

(cf. [**4**, Theorem 3.9ff.]), yet by suitable choice of $\{\eta_j\}_1^\infty$ we may ensure that the set of limit points of $\{\gamma_{3^n}(d\alpha)^{1/3^n}\}_1^\infty$ is any closed subinterval of $[0, 2]$. Ullman and Wyneken [**11**, **13**] (see also the references in [**4**]) have studied this and related phenomena, considering both the case where all carriers of $d\alpha$ (that is, all measurable subsets of the support of $d\alpha$ having full $d\alpha$ measure) have positive logarithmic capacity, and where some have zero capacity. Of course, $d\alpha$ of (1.2) to (1.4) has a carrier that is just a countable set.

The proofs of Theorems 1.1 to 1.3 are contained in §2. Finally, we remark that the above construction may be modified to yield pure jump distributions on **R**, whose contracted zero distribution is the same as that of $\exp(-|x|^\alpha)$, $\alpha > 0$.

NOTE ADDED IN PROOF. The author thanks W. Van Assche for pointing out that the proof of Theorem 1.3(a) yields $1/\sqrt{3}$, not $1/\sqrt{2}$ as originally stated. Al. Magnus has given a proof of Theorem 1.3 using only real variable methods. See *Sieved orthogonal polynomials and discrete measures with jumps in an interval*, by Van Assche and Magnus, to appear in this journal.

2. Proofs. Throughout, we assume that $d\alpha(x)$ is given by (1.2) to (1.4). We shall need the associated measure defined on the circle, as follows:

$$\mu(\theta) := \begin{cases} \alpha(1) - \alpha(\cos\theta), & \theta \in [0,\pi], \\ \alpha(\cos\theta) - \alpha(1), & \theta \in [-\pi, 0]. \end{cases} \tag{2.1}$$

Thus, if $d\alpha$ has a jump of size m at $x = \cos\theta$, then $\mu(\theta)$ has a jump of size m at $\pm\theta$. For $k = 1, 2, 3, \ldots$, let

$$u_k := e^{2\pi i/k}, \tag{2.2}$$

and define the finite sum

$$J_k[f] := \frac{1}{k}\sum_{j=0}^{k-1} f(u_k^{j+1/2}). \tag{2.3}$$

We see from (1.2) to (1.4), that for any f continuous on the unit circle,

$$\int_{-\pi}^{\pi} f(z)\, d\mu(\theta) = 2\sum_{j=1}^{\infty} \eta_j J_{2.3^j}[f], \qquad z := e^{i\theta}. \tag{2.4}$$

Associated with $d\mu(\theta)$ are the orthonormal polynomials

$$\varphi_n(d\mu, z) := \kappa_n(d\mu)z^n + \cdots, \qquad \kappa_n(d\mu) > 0,\ n = 0, 1, 2, \ldots,$$

satisfying

$$(2\pi)^{-1}\int_{-\pi}^{\pi} \varphi_n(d\mu, z)\overline{\varphi_m(d\mu, z)}\, d\mu(\theta) = \delta_{mn}, \qquad z := e^{i\theta}.$$

It is known [**2**, p. 194] that

$$\kappa_n(d\mu)^{-2} = \min_{\substack{\deg(P)=n \\ P\ \text{monic}}} (2\pi)^{-1}\int_{-\pi}^{\pi} |P(z)|^2\, d\mu(\theta), \qquad z := e^{i\theta}, \tag{2.5}$$

and that

$$\kappa_n(d\mu)^2 - \kappa_{n-1}(d\mu)^2 = |\varphi_n(d\mu, 0)|^2, \qquad n = 1, 2, 3, \ldots. \tag{2.6}$$

The relationship between $\kappa_n(d\mu)$ and $\gamma_n(d\alpha)$ is given by [**10**, p. 294]

$$\gamma_n(d\alpha) = (2\pi)^{-1/2}2^n\kappa_{2n}(d\mu)\{1 + \varphi_{2n}(d\mu, 0)/\kappa_{2n}(d\mu)\}^{1/2}, \qquad n = 0, 1, 2, \ldots. \tag{2.7}$$

Finally, we note that [**10**, p. 28],

$$\gamma_n(d\alpha)^{-2} = \min_{\substack{\deg(P)=n \\ P\ \text{monic}}} \int_{-1}^{1} P^2(x)\, d\alpha(x). \tag{2.8}$$

PROOF OF THEOREM 1.3(a). Choose a positive integer l such that

$$r := 3^{l-1} \le n < 3^l. \tag{2.9}$$

Now, if $n > r$,

$$P(x) := (2^{1-r}T_r(x))(2^{1-(n-r)}T_{n-r}(x))$$

is a monic polynomial of degree n. So, from (2.8),

$$\gamma_n(d\alpha)^{-2} \le 2^{2(2-n)} \int_{-1}^{1} (T_r(x)T_{n-r}(x))^2 \, d\alpha(x) = 2^{2(2-n)}\pi^{-1} \sum_{j=l}^{\infty} \eta_j I_{3^j}[(T_r T_{n-r})^2],$$

since all abscissas of I_{3^j}, $j = 1, 2, \ldots, l-1$, are zeros of T_r. This property, incidentally, explains our choice of powers of 3. Exactness of the Gauss quadrature rule yields, for $j \ge l$,

$$I_{3^j}[(T_r T_{n-r})^2] = \int_{-1}^{1} (T_r(x)T_{n-r}(x))^2 (1-x^2)^{-1/2} \, dx = \begin{cases} \frac{\pi}{4}, n \ne 2r, \\ \frac{3\pi}{8}, n = 2r, \end{cases}$$

so that

$$\gamma_n(d\alpha)^{-2} \le \frac{3}{2} 2^{2(1-n)} \sum_{j=l}^{\infty} \eta_j,$$

which yields the left inequality in (1.9) if $n > r$. When $n = r$, we use $P_n(x) := 2^{1-n}T_n(x)$ to obtain similarly

$$\gamma_n(d\alpha)^{-2} \le 2^{1-2n} \sum_{j=l}^{\infty} \eta_j. \tag{2.10}$$

For the right inequality in (1.9), we note that for any monic orthogonal polynomial $P(x)$ of degree n, where n satisfies (2.9), we have

$$\int_{-1}^{1} P^2(x) \, d\alpha(x) \ge \pi^{-1} \sum_{j=l}^{\infty} \eta_j I_{3^j}[P^2] = \pi^{-1} \left(\sum_{j=l}^{\infty} \eta_j \right) \int_{-1}^{1} P^2(x)(1-x^2)^{-1/2} \, dx,$$

by the exactness of the Gauss quadrature rules. Since $2^{1-n}T_n(x)$ is the unique monic polynomial minimizing this last right-hand side [**10**, p. 28], we obtain

$$\gamma_n(d\alpha)^{-2} \ge 2^{1-2n} \left(\sum_{j=l}^{\infty} \eta_j \right),$$

and hence the right inequality in (1.9). When $n = 3^{l-1}$, we obtain from (2.10) and the last inequality that there is equality in the right-hand side of (1.9). Then also, $2^{1-n}T_n(x)$ is the monic orthogonal polynomial of degree n for $d\alpha$. □

Next, we turn our attention to $\kappa_n(d\mu)$, but first, we need a lemma on J_k (see (2.3)). We use the notation $k|l$ to denote that k divides l.

LEMMA 2.2. *Let k be a positive integer. Then for any polynomial*

$$P(z) := \sum_{j=0}^{m} c_j z^j, \qquad J_k[P] = \sum_{k|j} c_j(-1)^{j/k}, \tag{2.11}$$

and, if $k > m$,

$$J_k[|P|^2] = (2\pi)^{-1} \int_{-\pi}^{\pi} |P(z)|^2 \, d\theta = \sum_{j=0}^{m} |c_j|^2. \tag{2.12}$$

PROOF. From (2.2) and (2.3), for any integer l,

$$\begin{aligned} J_k[z^l] &= u_k^{l/2} k^{-1} \sum_{j=0}^{k-1} u_k^{lj}, \\ &= \begin{cases} u_k^{l/2} k^{-1}\{u_k^{lk} - 1\}/\{u_k^l - 1\}, & u_k^l \neq 1 \\ u_k^{l/2}, & u_k^l = 1 \end{cases} = \begin{cases} 0, & u_k^l \neq 1, \\ (-1)^{l/k}, & u_k^l = 1, \end{cases} \end{aligned}$$

since $u_k^{lk} = 1$. Then (2.11) and (2.12) follow easily. □

We remark at this stage that although we have an exact formula for $\gamma_n(d\alpha)$ and $p_n(d\alpha, x)$ for n a positive integer power of 3, we cannot immediately deduce explicit corresponding formulas for $\kappa_{2n}(d\mu)$ and $\varphi_{2n}(d\mu, z)$ (cf. [**10**, p. 295, last paragraph]). If this were possible, our subsequent analysis could be simplified.

LEMMA 2.3. *Let* $l \geq 3$ *and* n *be a positive integer satisfying*

$$2 \cdot 3^{l-1} \leq n < 2 \cdot 3^l. \tag{2.13}$$

Let $r < l - 1$ *be a positive integer. Then*

$$1 \leq \kappa_n(d\mu)^{-2} \Big/ \left\{ \pi^{-1} \sum_{j=l}^{\infty} \eta_j \right\} \leq 3^{r-l+1} + \left\{ \sum_{j=r+1}^{\infty} \eta_j \right\} \Big/ \left\{ \sum_{j=l}^{\infty} \eta_j \right\}. \tag{2.14}$$

PROOF. Firstly, by Lemma 2.2, and by (2.4), and (2.5),

$$\begin{aligned} \kappa_n(d\mu)^{-2} &\geq (2\pi)^{-1} 2 \min_{\substack{\deg(P)=n \\ P \text{ monic}}} \sum_{j=l}^{\infty} \eta_j J_{2.3^j}[|P|^2] \\ &= \pi^{-1} \left(\sum_{j=l}^{\infty} \eta_j \right) \min_{\substack{\deg(P)=n \\ P \text{ monic}}} (2\pi)^{-1} \int_{-\pi}^{\pi} |P(z)|^2 \, d\theta \\ &= \pi^{-1} \left(\sum_{j=l}^{\infty} \eta_j \right), \end{aligned}$$

since [**10**] $P(z) := z^n$ is the monic polynomial minimizing the last integral. Hence we obtain the left inequality in (2.14). Next, let

$$N := 2 \cdot 3^{l-1}, \quad R := 2 \cdot 3^r, \quad S := 3^{l-1-r}. \tag{2.15}$$

Further, let

$$Q(z) := \sum_{t=0}^{S-1} (-z^R)^t, \tag{2.16}$$

and

$$P(z) := z^{n-N}\{z^N + Q(z)/S\}. \tag{2.17}$$

Note that $Q(z)$ has degree $(S-1)R < SR = N$, and hence $P(z)$ is a monic polynomial of degree n. Now, if

$$z := u_{2\cdot 3^j}^{s+1/2} = \exp\left(2\pi i \frac{s+1/2}{2\cdot 3^j}\right), \quad s = 0, 1, 2, \ldots, 2\cdot 3^j - 1, \tag{2.18}$$

some $j \le l-1$, then

$$z^N = \exp(\pi i(2s+1)3^{l-1-j}) = -1.$$

Hence, if $z^R \neq -1$, we have

$$Q(z) = \frac{1-(-z^R)^S}{1-(-z^R)} = \frac{1+z^N}{1+z^R} = 0,$$

and so $|P(z)| = 1$. If $z^R = -1$, we have $Q(z) = S$, and so $P(z) = 0$. Thus

$$J_{2\cdot 3^j}[|P|^2] \le J_{2\cdot 3^j}[1] = 1, \quad j \le l-1. \tag{2.19}$$

Further, if z is given by (2.18) for some $j \le r$, then

$$z^R = \exp(\pi i(2s+1)3^{r-j}) = -1,$$

so that $P(z) = 0$. Hence

$$J_{2\cdot 3^j}[|P|^2] = 0, \qquad j \le r. \tag{2.20}$$

Then

$$\begin{aligned}
\kappa_n(d\mu)^{-2} &\le (2\pi)^{-1}\int_{-\pi}^{\pi} |P(z)|^2\, d\mu(\theta) \\
&\le \pi^{-1}\sum_{j=r+1}^{l-1}\eta_j + \pi^{-1}\left(\sum_{j=l}^{\infty}\eta_j\right)(2\pi)^{-1}\int_{-\pi}^{\pi}|P(z)|^2\, d\theta \\
&= \pi^{-1}\sum_{j=r+1}^{l-1}\eta_j + \pi^{-1}\left(\sum_{j=l}^{\infty}\eta_j\right)\{1+S^{-1}\},
\end{aligned}$$

since the leading coefficient of $P(z)$ is 1, and the remaining S nonzero coefficients equal $1/S$. The second inequality in (2.14) then follows. □

PROOF OF THEOREM 1.3(b). We first use Lemma 2.3 to show that

$$\lim_{n\to\infty} \kappa_{2n}(d\mu)^{-2}\Big/\left\{\pi^{-1}\sum_{j=\langle\log_3 n\rangle+1}^{\infty}\eta_j\right\} = 1. \tag{2.21}$$

Now (1.10) implies that

$$\lim_{l\to\infty}\sum_{j=l}^{\infty}\eta_j\Big/\sum_{j=l+1}^{\infty}\eta_j = 1.$$

It follows that we can find a nondecreasing sequence of positive integers $\{a_j\}_1^\infty$ with limit ∞, such that

$$\lim_{l\to\infty}\sum_{j=l-a_l}^{\infty}\eta_j\Big/\sum_{j=l}^{\infty}\eta_j = 1.$$

Writing $r := r(l) := l - a_l - 1$, we have

$$\lim_{l\to\infty} \sum_{j=r+1}^{\infty} \eta_j \Big/ \sum_{j=l}^{\infty} \eta_j = 1 \quad \text{and} \quad \lim_{l\to\infty} 3^{r-l+1} = 0.$$

Then Lemma 2.3 yields (2.21). Further, (2.6) yields

$$\begin{aligned}\lim_{n\to\infty} &\{|\varphi_{2n-1}(d\mu,0)|^2 + |\varphi_{2n}(d\mu,0)|^2\}/\kappa_{2n}(d\mu)^2 \\ &= \lim_{n\to\infty}\{1 - \kappa_{2n-2}(d\mu)^2/\kappa_{2n}(d\mu)^2\} = 0.\end{aligned}$$

Then (2.7) and (2.21) yield (1.11). □

PROOF OF THEOREM 1.1. Let

$$\eta_l := (2\xi_{3^{l-1}}^2)^{-1} - (2\xi_{3^l}^2)^{-1}, \qquad l \text{ large enough.}$$

It is easily seen that

$$\xi_n = \left\{ 2 \sum_{j=\langle \log_3 n\rangle+1}^{\infty} \eta_j \right\}^{-1/2}, \qquad n = 3^l,\ l \text{ large enough.}$$

We may choose the remaining finitely many η_j so that (1.3) is satisfied. Since (1.6) yields (1.10), Theorem 1.3 yields the result. □

PROOF OF THEOREM 1.2. Use (1.9) and choose the $\{\xi_j\}_1^\infty$ in an obvious way. □

What about the rate with which $A_n(d\alpha)$ may approach $1/2$ as $n \to \infty$? Now if

$$\sum_{n=1}^{\infty} \left| A_n(d\alpha) - \frac{1}{2} \right| < \infty,$$

then [8, p. 124] $d\alpha$ belongs to Szegö's class, while if

$$\sum_{n=1}^{\infty} |A_{n+1}(d\alpha) - A_n(d\alpha)| < \infty,$$

then $\alpha'(x)$ is positive and continuous in $(-1,1)$, but does not necessarily belong to Szegö's class [7, p. 612]. To see what sort of rates may be gleaned from Lemma 2.4, let us set

$$\eta_j := 1/\{j(j+1)\}, \qquad j = 1,2,3,\ldots,$$

so that

$$\sum_{j=n}^{\infty} \eta_j = \frac{1}{n}, \qquad n = 1,2,3,\ldots\ .$$

Choosing l large enough, and $r := l - 3\langle \log_3 l\rangle$ in Lemma 2.3, yields

$$1 \le \{\kappa_{2\cdot 3^l}(d\mu)/\kappa_{2\cdot 3^{l-1}}(d\mu)\}^{-2} \le l^{-2} + 1 + 4(\log_3 l)/l.$$

Defining

$$\phi_j(d\mu,z) := \varphi_j(d\mu,z)/\kappa_j(d\mu), \qquad j = 0,1,2,\ldots,$$

we obtain from (2.6), for some C independent of l,

$$\sum_{j=2\cdot 3^{l-1}+1}^{2\cdot 3^l} |\phi_j(d\mu,0)|^2 \le C(\log l)/l, \qquad l \text{ large enough.} \tag{2.22}$$

Since [**4**, (3.4), **8**, p. 129]

$$2A_n(d\alpha) = \{[1 + \phi_{2n-2}(d\mu, 0)][1 - \phi^2_{2n-1}(d\mu, 0)][1 - \phi_{2n}(d\mu, 0)]\}^{1/2}, \tag{2.23}$$

one can use (2.22) to show that for l large enough, and some C_1 independent of l, we have

$$\sum_{j=2\cdot 3^{l-1}+1}^{2\cdot 3^l} \left|A_j(d\alpha) - \frac{1}{2}\right|^2 \leq \frac{C_1(\log l)}{l}. \tag{2.24}$$

It seems quite likely that given any decreasing sequence $\{\varepsilon_n\}_1^\infty$ of positive numbers with

$$\sum_{n=1}^{\infty} \varepsilon_n = \infty,$$

there exists a pure jump distribution $d\alpha$, with

$$|A_n(d\alpha) - 1/2| < \varepsilon_n, \quad n \text{ large enough.}$$

References

1. F. Delyon, B. Simon and B. Souillard, *From power pure point to continuous spectrum in disordered systems*, Ann. Inst. H. Poincaré Phys. Theor. **42** (1985), 283–309.
2. G. Freud, *Orthogonal polynomials*, Akademiai Kiado/Pergamon Press, Budapest, 1971.
3. Ya. L. Geronimus, *Orthogonal polynomials*, Consultants Bureau, 1961.
4. D. S. Lubinsky, *A survey of general orthogonal polynomials for weights on finite and infinite intervals*, Acta Appl. Math. **10** (1987), 237–296.
5. Al. Magnus, *Toeplitz matrix techniques and convergence of complex weight Padé approximants*, J. Comput. Appl. Math. **19** (1987), 23–38.
6. ——, *Problem section*, Proceedings of the Segovia 1986 Conference on Orthogonal Polynomials.
7. A. Máté and P. Nevai, *Orthogonal polynomials and absolutely continuous measures*, Approximation Theory IV (C. K. Chui, et al., eds.), Academic Press, New York, 1983, pp. 611–617.
8. P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. no. 213, Amer. Math. Soc., Providence, R.I., 1979.
9. P. Nevai, *Geza Freud orthogonal polynomials and Christoffel functions, a case study*, J. Approx. Theory **48** (1986), 3–167.
10. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1939, 4th ed., 1975.
11. J. L. Ullman and M. F. Wyneken, *Weak limits of zeros of orthogonal polynomials*, Constr. Approx. **2** (1986), 339–347.
12. J. Von Neumann, *Charakterieserung des Spektrums eines Integraloperators*, Actualités Sci. Indust. no. 229, Hermann, Paris, 1935.
13. M. F. Wyneken, *Norm asymptotics of orthogonal polynomials for general measures*, Constr. Approx. **4** (1988), 123–131.

NATIONAL RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, P. O. BOX 395, PRETORIA 0001, REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, P. O. WITS 2050, REPUBLIC OF SOUTH AFRICA