PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

COMMENSURATE SEQUENCES OF CHARACTERS

A. PEŁCZYŃSKI

(Communicated by William J. Davis)

ABSTRACT. If (a_j) and (b_j) are sequences of characters on compact abelian groups S and T respectively such that for every sequence of scalars (α_j) $\|\sum \alpha_j a_j\|_\infty \asymp \|\sum \alpha_j b_j\|_\infty$ then for every $1 \le p < \infty$ and every sequence (x_j) of elements of an arbitrary Banach space X

$$\int_S \left\|\sum x_j a_j\right\|^p ds \asymp \int_T \left\|\sum x_j b\right\|^p dt.$$

This result generalizes a result of Pisier [**Pi 1**] for Sidon sets. For topological Sidon sets on $\mathbf{R}$ a slightly stronger result holds.

Let (f_j) and (g_j) be sequences of scalar-valued functions in $L^\infty(\mu)$ and $L^\infty(\nu)$ respectively (μ, ν-probability measures). Call (f_j) and (g_j) *d-commensurate* if there exists a $d \ge 1$ such that for every Banach space X and for every $p \in [1, \infty)$ and every sequence $(x_j) \subset X$,

$$(1) \qquad d^{-p} \int \left\|\sum_j f_j x_j\right\|^p d\mu \le \int \left\|\sum_j g_j x_j\right\|^p d\nu \le d^p \int \left\|\sum_j f_j x_j\right\|^p d\mu.$$

Pisier [**Pi 1**] observed that if (f_j) and (g_j) are Sidon sets of characters on compact abelian groups then they are commensurate. Using the technique which goes back to Rudin (cf. [**R**], Theorem 3.1) we generalize Pisier's result to the following.

THEOREM 1. *Let $A = \bigcup_j \{a_j\}$ and $B = \bigcup_j \{b_j\}$ be sets of characters on compact abelian groups S and T respectively. Assume that the correspondence $a_j \to b_j$ extends to a linear isomorphism, say U from $C_A(S)$ onto $C_B(T)$ such that $\|U\|\,\|U^{-1}\| = d$.*

Then the sequences (a_j) and (b_j) are d-commensurate.

Here by $C_\Lambda(G)$ we denote the subspace of continuous complex-valued functions on a compact abelian group G generated by characters from a fixed set Λ of the dual group $\widehat{G}$, i.e.

$$C_\Lambda(G) = \left\{ f \in C(G) \colon \hat{f}(\gamma) = \int_G f(x)\gamma(-x)\,dx = 0 \text{ for } \gamma \in \widehat{G} \backslash \Lambda \right\}.$$

Received by the editors May 28, 1987. Part of this paper has been presented to the American Math. Society in the special session on Functional Analysis at the meeting of AMS in Kent Ohio in the spring of 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46E40, 43A46, 43A15.

Key words and phrases. Commensurate sequences, Sidon sets.

This paper was written while the author participated in the Special Year in Modern Analysis at the University of Illinois. The author would like to thank the Mathematics Department of the University for its hospitality and support.

PROOF. Fix $t \in T$ and denote by δ_t the point mass at t. Let ϕ_t^* be the restriction of δ_t to $C_B(T)$. Clearly ϕ_t^* is a linear functional on $C_B(T)$ with $\|\phi_t^*\| \le 1$. Obviously for every $b_j \in B$,

$$b_j(t) = \phi_t^*(b_j) = \phi_t^*(U(a_j)) = (U^*\phi_t^*)(a_j).$$

Let μ_t be a complex Borel measure on S which via the Riesz Representation Theorem corresonds to the Hahn-Banach extension onto $C(S)$ of the linear functional $U^*\phi_t^*$ on $C_A(S)$. Define the measures ν_t and $\overline{\nu}_t$ by

$$\nu_t(V) = \mu_t(-V); \quad \overline{\nu}_t(V) = \overline{\mu_t(-V)} \qquad \text{for Borel } V \subset S.$$

Then the Fourier coefficients of ν_t at the points of A satisfy

$$\hat{\nu}_t(a_j) = \int_S a_j(-s)\nu_t(ds) = \int_S a_j(s)\mu_t(ds) = b_j(t),$$

clearly the total variation of ν_t satisfies

$$\|\overline{\nu}_t\| = \|\nu_t\| = \|\mu_t\| \le \|U^*\| = \|U\|.$$

Now for a fixed (eventually zero) sequence (x_j) of elements of a Banach space X define the X-valued functions f and f_t by

$$f = \sum_j x_j a_j; \quad f_t = \sum_j x_j b_j(t) a_j.$$

Let $f * \nu$ denote the convolution on S of a scalar measure ν with a vector-valued function f, i.e.

$$(f * \nu)(s) = \int_S f(s-\sigma)\nu(d\sigma) \quad \text{for } s \in S.$$

Then comparing the Fourier coefficients we get

$$f * \nu_t = f_t; \quad f_t * \overline{\nu}_t = f.$$

Thus, by the vector-valued Young inequality,

$$\|f_t\|_p \le \|f\|_p \|\nu_t\| \le \|U\|\,\|f\|_p; \quad \|f\|_p \le \|U\|\,\|f_t\|_p$$

where $\|f\|_p = (\int_S |f|^p\,ds)^{1/p}$. Integrating these inequalities against the Haar measure dt of T and using the Fubini Theorem (since the sequence (x_j) is eventually zero and the functions a_j and b_j are continuous, there is no problem of measurability and integrability) we get

$$\|U\|^{-1}\|f\|_p \le \left(\int_S \int_T \left\|\sum_j a_j(s)b_j(t)x_j\right\|^p ds\,dt\right)^{1/p} \le \|U\|\,\|f\|_p.$$

Exchanging the role of A and B for $g = \sum x_j b_j$ we get

$$\|U^{-1}\|^{-1}\|g\|_p \le \left(\int_S \int_T \left\|\sum_j a_j(s)b_j(t)x_j\right\|^p ds\,dt\right) \le \|U^{-1}\|\,\|g\|_p$$

thus

$$(\|U\|\,\|U^{-1}\|)^{-1}\|f\|_p \le \|g\|_p \le \|U\|\,\|U^{-1}\|\,\|f\|_p.$$

REMARK. Similarly as in Pisier's paper **[Pi 1]** the L^p-norms in (1) can be replaced by the "averages" $\int_S \phi\|\sum x_k a_k\|\,ds$ where $\phi\colon \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous nondecreasing convex function.

Obviously a sequence of characters which is commensurate with the Rademacher sequence forms a Sidon set. However in that case we have also the following criterion.

PROPOSITION 2. *If (a_k) is a sequence of characters on a compact abelian group S such that for some q with $1 \le q < \infty$ there exists a constant C such that for every finite sequence $(x_k)_{1\le k\le N}$ $(N = 1, 2, \dots)$ of elements of every L^p space with $2 < p \le \infty$.*

$$\int_S \left\|\sum_{k=1}^N a_k(s)x_k\right\|_{L^p}^q ds \le C\int_0^1 \left\|\sum_{k=1}^N r_k(w)x_k\right\|_{L^p}^q dw \tag{2}$$

then $\bigcup_k\{a_k\}$ is a Sidon set.

PROOF. Fix $p > 2$. Specify L^p to be $L^p(S)$. Fix a sequence of scalars (α_k) and put $x_k = \alpha_k a_k$ $(k = 1, 2, \dots)$. Then using the properties of the normalized Haar measure "ds" we get

$$\begin{aligned}\int_S \left\|\sum_{k=1}^N a_k(s)\alpha_k a_k\right\|_{L^p(S)}^q ds &= \int_S \left(\int_S \left|\sum_{k=1}^N \alpha_k a_k(s+t)\right|^p dt\right)^{q/p} ds\\ &= \int_S \left(\int_S \left|\sum_{k=1}^N \alpha_k a_k(t)\right|^p dt\right)^{q/p} ds\\ &= \left(\int_S \left|\sum_{k=1}^N \alpha_k a_k(t)\right|^p dt\right)^{q/p}.\end{aligned}$$

Observe that for $p > 2$,

$$\left(\int_S \left|\sum_{k=1}^N \alpha_k a_k(t)\right|^p dt\right)^{1/p} \ge \left(\sum_{k=1}^N |\alpha_k|^2\right).$$

On the other hand, by (2), we get

$$\begin{aligned}\left(\int_S \left|\sum_{k=1}^N \alpha_k a_k(t)\right|^p dt\right)^{q/p} &\le C\int_0^1 \left(\int_S \left|\sum_{k=1}^N \alpha_k a_k(t) r_k(w)\right|^p dt\right)^{q/p} dw\\ \text{(a)}\qquad &\le CC_q\left(\int_0^1 \left(\int_S \left|\sum_{k=1}^N \alpha_k a_k(t) r_k(w)\right|^p dt\right)^{1/p} dw\right)^q\\ \text{(b)}\qquad &\le CC_qB_p^q\left(\sum_{k=1}^N \left(\int_S |\alpha_k a_k(t)|^p\right)^{2/p} dt\right)^{q/2}\\ &= CC_qB_p^q\left(\sum_{k=1}^N |\alpha_k|^2\right)^{q/2}\end{aligned}$$

where (a) follows from the Kahane-Khinchine inequality, [**Kh**, p. 20], and (b) follows from the Orlicz inequaliity [**O**]. Moreover it is well known that the constant B_p appearing the Orlicz inequality is dominated by the constant in the Khinchine inequality between L_2 and L_p norms and therefore $B_p \leq c\sqrt{p}$ (cf. [**O**, and **Z**, Chapter V, §8, Theorem 8.4] thus we have just shown that there exists a numerical constant B such that

$$\left(\sum |\alpha_k|^2\right)^{1/2} \leq \left(\int_S \left|\sum \alpha_k a_k(t)\right|^p dt\right)^{1/p} \leq B\sqrt{p}\left(\sum |\alpha_k|^2\right)^{1/2}$$

for all scalars (α_k) and every $p > 2$. Hence, by a result of Pisier [**Pi 2**, or **Pi 3**], $\bigcup_k \{\alpha_k\}$ is a Sidon set.

Our next result concerns topological Sidon sets on the real line $\mathbf{R}$. Recall (cf. [**M**, p. 183] that an increasing sequence of positive real numbers, say (λ_k) is called a topological Sidon set if there exist a compact set $K \subset \mathbf{R}$ and $C > 0$ such that for every sequence of scalars (α_k)

$$C^{-1} \sum |\alpha_k| \leq \sup_{t \in K} \left|\sum \alpha_k e^{i\lambda_k t}\right|.$$

An important example of a topoplogical Sidon set is an increasing sequence (λ'_k) of positive reals satisfying the Hadamard condition $\inf_k \lambda_{k+1}\lambda_k^{-1} \geq q > 1$ (cf. [**M**, p. 185].

Our next result shows that if (λ_k) is a topological Sidon set then the functions $(e^{i\lambda_k t})$ considered on any bounded interval $[a, b] \subset R$ are commensurate with the Rademacher functions (r_k).

THEOREM 3. *Let $(\lambda_k) \subset \mathbf{R}$ be a topological Sidon set. Then for every interval $[a, b]$ with $-\infty < a < b < +\infty$ there exists a constant $C = C(b-a)$ such that for every p with $1 \leq p < \infty$ and every sequence (x_j) of elements of an arbitrary Banach space X*

$$C^{-p} \int_0^1 \left\|\sum r_k(w)x_k\right\|^p dw \leq \int_a^b \left\|\sum e^{i\lambda_k t} x_k\right\|^p dt \leq C^p \int_0^1 \left\|\sum r_k(w)x_k\right\|^p dw.$$

PROOF. Since (λ_k) is a topological Sidon set, for every δ with $b - a > 2\delta > 0$ there is a constant $C = C(\delta)$ such that for every scalars $\alpha_1, \alpha_2, \ldots$

$$\sum |\alpha_k| \leq C \sup_{|t| \leq \delta} \left|\sum_k \alpha_k e^{i\lambda_k t}\right| \tag{3}$$

(cf. [**DG 1** or **M**, p. 194, Theorem VII]).

Fix a positive integer N. Let $\varepsilon = (\varepsilon_k)_{1 \leq k \leq N}$ be a sequence of ± 1 of length N. It follows from (3) that there exists on the linear span of $(e^{i\lambda_k t})_{1 \leq k \leq N}$ regarded as a subspace of $C([-\delta, \delta])$ a linear functional of norm $\leq C$ which takes the value ε_k at $e^{i\lambda_k t}$ for $k = 1, 2, \ldots, N$. Thus using the Hahn-Banach and the Riesz Representation Theorems we infer that there is a complex Borel measure μ on $\mathbf{R}$ such that $\|\mu\| \leq C$, μ is concentrated on $[-\delta, \delta]$, $\int_{-\delta}^{\delta} e^{i\lambda_k s}\mu(ds) = \varepsilon_k$ for $k = 1, 2, \ldots, N$. Let $|\mu|$ be the unique nonnegative Borel measure on $\mathbf{R}$ such that μ is absolutely continuous with respect to $|\mu|$ and the Radon-Nikodym derivative $d\mu/d|\mu|$ is a unimodular function $|\mu|$-almost everywhere.

Now fix a sequence $(x_k)_{1\le k\le N}$ in X and define functions $f\colon \mathbf{R}\to X$ and $f_\varepsilon\colon \mathbf{R}\to X$ by

$$f(t)=\sum_{k=1}^{N} e^{i\lambda_k t}x_k;\quad f_\varepsilon(t)=\sum_{k=1}^{N}\varepsilon_k e^{i\lambda_k t}x_k \qquad (t\in\mathbf{R}).$$

Then for every $t\in\mathbf{R}$

$$f(t)=\sum_{k=1}^{N}\varepsilon_k e^{i\lambda_k t}\int_{-\delta}^{\delta} e^{i\lambda_k s}\mu(ds)=\int_{-\delta}^{\delta} f_\varepsilon(t+s)\mu(ds).$$

Hence remembering that $\|\,|\mu|\,\|=|\mu|([-\delta,\delta])\le C$ and using the Hölder inequality we get

$$\|f(t)\|^p\le C^{p-1}\int_{-\delta}^{\delta}\|f(t+s)\|^p|\mu|(ds)\qquad (1\le p\le\infty).$$

Integrating against the Lebesgue measure and using the Fubini Thoerem we get

$$\begin{aligned}\int_a^b\|f(t)\|^p\,dt&\le C^{p-1}\int_a^b\int_{-\delta}^{\delta}\|f_\varepsilon(t+s)\|^p|\mu|(ds)\,dt\\ &=C^{p-1}\int_{-\delta}^{\delta}\left(\int_{a+s}^{b+s}\|f_\varepsilon(u)\|^p\,du\right)|\mu|(ds)\\ &\le C^{p-1}\int_{-\delta}^{\delta}\left(\int_{a-\delta}^{b+\delta}\|f_\varepsilon(u)\|^p\,du\right)|\mu|(ds)\\ &\le C^p\int_{a-\delta}^{b+\delta}\|f_\varepsilon(u)\|^p\,du.\end{aligned}\tag{4}$$

Averaging over all sequences ε of ±1 of length N and using the identity

$$\operatorname*{Average}_{\varepsilon}\|f_\varepsilon(u)\|^p=\int_0^1\left\|\sum_{k=1}^{N}r_k(w)e^{i\lambda_k u}x_k\right\|^p dw,$$

and the Fubini Theorem we obtain

$$\int_a^b\|f(t)\|^p\,dt\le C^p\int_{a-\delta}^{b+\delta}\int_0^1\left\|\sum_{k=1}^{N}r_k(w)e^{i\lambda_k u}x_k\right\|^p dw\,du.$$

By the principle of contraction (cf. [**Kh**, p. 21]) for unimodular complex numbers $(e^{i\lambda_k u})_{1\le k\le N}$ we get

$$\int_0^1\left\|\sum_{k=1}^{N}r_k(w)e^{i\lambda_k u}x_k\right\|^p dw\le(\sqrt{2})^p\int_0^1\left\|\sum_{k=1}^{N}r_k(w)x_k\right\|^p dw.$$

Thus

$$\int_a^b\|f(t)\|^p\,dt\le(\sqrt{2}C)^p(b-a+2\delta)\int_0^1\left\|\sum_{k=1}^{N}r_k(w)x_k\right\|^p dw.$$

To prove the estimate in the opposite direction we replace the interval $[a,b]$ by $[a+\delta,b-\delta]$ and f by f_ε. Then we get as in (4)

$$\int_{a+\delta}^{b-\delta}\|f_\varepsilon(t)\|^p\,dt\le C^p\int_a^b\|f(u)\|^p\,du.$$

Averaging over all sequences of ± 1 of length N we get

$$\int_{a+\delta}^{b-\delta}\int_0^1 \left\|\sum_{k=1}^N r_k(w)e^{i\lambda_k t}x_k\right\|^p dw\,dt \le C^p\int_a^b \|f(u)\|^p\,du. \tag{5}$$

Next for $t\in[a+\delta,b-\delta]$ put

$$g_t=\prod_{k=1}^N(1+r_k\cos\lambda_k t)+i\prod_{k=1}^N(1-r_k\sin\lambda_k t).$$

Then $\operatorname{Re} g_t\ge 0$, $\operatorname{Im} g_t\ge 0$ and $\int_0^1\operatorname{Re} g_t\,dw=\int_0^1\operatorname{Im} g_t\,dw=1$ thus $\int_0^1|g_t(w)|\,dw\le 2$. Moreover we have

$$\int_0^1\left(\sum_{k=1}^N r_k(w\dotplus w')e^{i\lambda_k t}x_k\right)g_t(w')\,dw'=\sum_{k=1}^N r_k(w)e^{i\lambda_k t}x_k$$

where $\dotplus$ denotes the multiplication on the Diadic Group $\{1,-1\}^\infty$ transported to $[0,1]$ via the measure preserving map $(\varepsilon_j)_{j\ge1}\to\sum_{j=1}^\infty 2^{-j-1}(1-\varepsilon_j)$. Thus, by the Young inequality,

$$\int_0^1\left\|\sum_{k=1}^N r_k(w)x_k\right\|^p dw\le 2^p\int_0^1\left\|\sum_{k=1}^N r_k(w)e^{i\lambda_k t x_k}\right\|^p dw.$$

Thus integrating over $[a+\delta,b-\delta]$ and using (5) we get

$$(2C)^{-p}(b-a-2\delta)\int_0^1\left\|\sum_{k=1}^N r_k(w)x_k\right\|^p dw\le\int_a^b\|f(t)\|^p\,dt.$$

REMARK. Theorem 3 can be generalized to arbitrary topological Sidon sets in locally compact abelian groups studied in [**DG 2**].

ADKNOWLEDGMENT. The author is indebted to R. Kaufman for supplying the main ingredient in the proof of Theorem 3 and conjecturing Proposition 2. A slightly weaker fact than Theorem 3 has also been proved by O. Blasco.

REFERENCES

[**DG 1**] M. Deschamps-Godim, *Sur les ensembles de Sidon topologiques*, C. R. Acad. Sci. Paris **271** (1970), 1247–1249.

[**Dg 2**] ——, *Ensembles de Sidon topologiques*, Ann. Inst. Fourier (Grenoble) **22** (1972), 51–79.

[**Kh**] J.-P. Kahane, *Some random series of functions*, Cambridge Univ. Press, 1985.

[**M**] Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland, 1972.

[**O**] W. Orlicz, *Über unbedingte Konvergenz in Funktionräumen.* I, II, Studia Math **4** (1933), 33–37, 41–47.

[**Pi 1**] G. Pisier, *Les inégalités de Khinchine-Kahane d'après C. Borel*, Séminaire sur le Géométrie des Espaces de Banach 1977–1978, Expose VII, Ecole Polytechnique, Centre de Mathématiques, 1978.

[**Pi 2**] ——, *Ensembles de Sidon et processes gaussiens*, C. R. Acad. Sci. Paris Sér. A **286** (1978), 671–674.

[**Pi 3**] ——, *De nouvelles caractérisation des ensembles de Sidon*, Math Anal. Appl., Part B, Advances in Math. Suppl. Studies **7B** (1981).

[**R**] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227.

[**Z**] A. Zygmund, *Trigonometric series.* I, Cambridge Univ. Press, 1959.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, SNIADECKICH 8, IP, 00-950 WARSZAWA, POLAND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN STREET, URBANA, ILLINOIS 61801