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SUBSPACES OF $L_{p,q}$

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ABSTRACT. We examine the subspace structure of the Lorentz function space $L_{p,q}[0,\infty)$. Our main result is that a subspace of $L_{p,q}[0,\infty)$, $p \neq 2$, $q < \infty$, must either strongly embed in $L_p[0,1]$ or contain a complemented copy of l_q.

1. Introduction. In this paper we examine the subspace structure of the Lorentz function spaces $L_{p,q}[0,1]$ and $L_{p,q}[0,\infty)$. (Throughout, *subspace* will mean a closed, infinite-dimensional subspace.) Our main result, presented in §2, is that a subspace of $L_{p,q}[0,\infty)$, $p \neq 2$, $q < \infty$, is either isomorphic to a strongly embedded subspace of $L_p[0,1]$, or contains a complemented copy of l_q. (Recall that a subspace X of L_p is *strongly embedded* if the L_p- and L_0-topologies on X coincide.)

From one point of view our results should be considered as extensions of certain results which are known for $L_{p,q}[0,1]$ and $l_{p,q}$ (cf., e.g., [**9**, **1**]). But, unlike the space L_p, results for $L_{p,q}[0,1]$ do not automatically extend to the space $L_{p,q}[0,\infty)$; indeed, as we shall see, the spaces $L_{p,q}[0,1]$ and $L_{p,q}[0,\infty)$ are not isomorphic. Thus we prefer to emphasize a slightly different point of view: the interval $[0,\infty)$ is the more natural setting and plays an essential role in our arguments. Moreover, several of our techniques are quite general and may be of independent interest (in particular, Corollary 2.7 below).

Our notation is more or less standard and for the most part agrees with that of [**20**]. In what follows we will need to make use of many facts about $L_{p,q}$, both as a lattice and as an interpolation space; while we will attempt to recall most of these facts, the reader is encouraged to refer to [**20**] and its references for any unexplained terminology.

We write the Lebesgue measure of a measurable subset A of $\mathbf{R}^n$ as $|A|$. Given a measurable, real-valued function f we define the *support* of f as $\operatorname{supp} f = \{f \neq 0\}$, the *distribution* of f as $d_f(t) = |\{|f| > t\}|$, and the *decreasing rearrangement* of f as $f^*(t) = \inf\{s > 0 \colon d_f(s) \leq t\} = d_f^{-1}(t)$. (Note that d_f is actually the distribution of $|f|$ and that $f^*\colon [0,\infty) \to \mathbf{R}$.)

For $0 < p < \infty$, $0 < q \leq \infty$, and $I = [0,1]$ or $[0,\infty)$, the Lorentz function space $L_{p,q}(I)$ is the space (of equivalence classes) of all measurable functions f on I for

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which $\|f\|_{p,q} < \infty$, where

$$
(1) \qquad \begin{aligned} \|f\|_{p,q} &= \left(\int_I f^*(t)^q \, d(t^{q/p}) \right)^{1/q}, \qquad && q < \infty, \\ &= \sup_{t \in I} t^{1/p} f^*(t), && q = \infty. \end{aligned}
$$

If the interval I is clear from the context, or if a particular discussion does not depend on I, we will simply write $L_{p,q}$.

There is also an obvious sequence space analogue of $L_{p,q}$: the space $l_{p,q}$ of all sequences $(a_i)_{i=1}^{\infty}$ for which $\|(a_i)\|_{p,q} < \infty$, where

$$
(2) \qquad \begin{aligned} \|(a_i)\|_{p,q} &= \left\{ \sum_{i=1}^{\infty} a_i^{*q} (i^{q/p} - (i-1)^{q/p}) \right\}^{1/q}, \qquad && q < \infty, \\ &= \sup_i i^{1/p} a_i^*, && q = \infty, \end{aligned}
$$

and where (a_i^*) is the decreasing rearrangement of $(|a_i|)$. Clearly, $l_{p,q}$ is isometric to a sublattice of $L_{p,q}[0,\infty)$. Also, for any p we have $L_{p,p} = L_p$ and $l_{p,p} = l_p$; in this case we will simply write $\|\cdot\|_p$. It is known that these norms satisfy $\|f\|_{p,q_2} \leq \|f\|_{p,q_1}$ whenever $q_1 < q_2$ [**20**, Proposition 2.b.9].

It is well known that for $1 \leq q \leq p < \infty$, (1) defines a norm under which $L_{p,q}$ is a separable, rearrangement invariant (r.i.) Banach function space; otherwise, (1) defines a quasi-norm on $L_{p,q}$ (which is known to be equivalent to a norm if $1 < p < q \leq \infty$). Because we are only concerned with the isomorphic structure of $L_{p,q}$ here, we will make *two conventions*: throughout we will simply refer to the expression in (1) as the "norm" on $L_{p,q}$; and we will use C as a symbol representing a positive, finite constant (whose precise value may change from line to line) which depends only on p and q.

Next recall that for any $0 < p < \infty$ and $0 < q \leq \infty$, $L_{p,q}$ is equal, up to an equivalent norm, to the space $[L_{p_1}, L_{p_2}]_{\theta,q}$ constructed using the *real interpolation method*, where $0 < p_1 < p_2 \leq \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_1 + \theta/p_2$. (See [**2** or **20**, Theorem 2.g.18].)

As a final bit of preliminary information we recall a result from [**6**]: $L_{p,q}$ satisfies an upper r-estimate and a lower s-estimate for disjoint elements where $r = \min(p,q)$ and $s = \max(p,q)$.

2. Subspaces of $L_{p,q}$.

We begin with a criterion for a sequence in $L_{p,q}$ to be equivalent to the l_q basis (cf. [**19**, Proposition 4.e.3]).

LEMMA 2.1. *Let $1 < p < \infty$, $1 \leq q < \infty$, and let (f_n) be a sequence of norm-one vectors in $L_{p,q}[0,\infty)$. If $f_n^* \to 0$ a.e. as $n \to \infty$, then there is a subsequence of (f_n) which is equivalent to the unit vector basis for l_q and which spans a complemented subspace of $L_{p,q}[0,\infty)$.*

PROOF. By a standard argument, we may choose a sequence $\varepsilon_n \downarrow 0$ so that $|\{|f_n| \geq \varepsilon_n\}| < \varepsilon_n$. Setting $U_n = \{|f_n| \geq \varepsilon_n\}$, $g_n = f_n \cdot \chi_{U_n}$, and $h_n = f_n - g_n$, we have $|\operatorname{supp} g_n| < \varepsilon_n$ and $|h_n| < \varepsilon_n$ a.e. Without loss of generality, we may also suppose that $\inf_n \|g_n\|_{p,q} > 0$ and $\inf_n \|h_n\|_{p,q} > 0$.

If $\varepsilon_n \to 0$ fast enough, then each of (g_n) and (h_n) will have a common subsequence equivalent to the l_q basis; the proof in either case follows more or less directly from the analogous result for $l_{p,q}$ (see [**18**, **1**, and **5**]). That this is so for (g_n) is due to Figiel, Johnson and Tzafriri [**9**] in case $q \leq p$, but the proof for $p < q$ is very similar (cf. e.g. [**3**]). The argument for (h_n) is again very similar; for completeness we will sketch the proof in the case $p < q$ (recall that in this case $\|\cdot\|_{p,q}$ is a quasi-norm with, say, constant C).

First, for simplicity, we may suppose that each h_n has norm one, has compact support A_n, and that the sequence $s_n = |A_n|$ is increasing. Now let $\varepsilon > 0$ and note that since $h_n \to 0$ uniformly we may further suppose, by passing to a subsequence if necessary, that

$$\|h_{n+1}\cdot\chi_A\|_{p,q} < \varepsilon/2^n C \quad \text{whenever } |A| < s_n. \tag{3}$$

Thus we may choose an automorphism τ of $[0,\infty)$ which, for each n, satisfies

$$B_{n+1} = \tau[s_n, s_{n+1}) \subset A_{n+1}, \qquad |A_{n+1}\backslash B_{n+1}| < s_n, \tag{4}$$

and

$$\int_{s_n}^{s_{n+1}} |h_{n+1}(\tau(t))|^q \, d(t^{q/p}) < 1+\varepsilon. \tag{5}$$

It now follows easily from (3), (4), (5), and the fact that $p < q$, that for any scalars (a_n) we have

$$\begin{aligned}
C^{-1}(1-\varepsilon)\left(\sum_n |a_n|^q\right)^{1/q} &\leq \left(\sum_n |a_n|^q \|h_n\chi_{B_n}\|_{p,q}^q\right)^{1/q} \\
&\leq \left\|\sum_n a_n h_n \chi_{B_n}\right\|_{p,q} \leq \left\|\sum_n a_n h_n\right\|_{p,q} \\
&\leq C\left(\sum_n |a_n|\,\|h_n\chi_{A_n\backslash B_n}\|_{p,q} + \left\|\sum_n a_n h_n\chi_{B_n}\right\|_{p,q}\right) \\
&\leq C\left\{\varepsilon\cdot\sum_n |a_n|\cdot 2^{-n} + \left(\sum_n |a_n|^q \int_{s_{n-1}}^{s_n} |h_n(\tau(t))|^q \, d(t^{q/p})\right)^{1/q}\right\} \\
&\leq C(1+2\varepsilon)\cdot\left(\sum_n |a_n|^q\right)^{1/q}.
\end{aligned}$$

Finally, the existence of a projection onto either $[g_n]$ or $[h_n]$ is straightforward (again see [**9**], but see also [**17**]). ■

COROLLARY 2.2. (i) *For $1 < p < \infty$, $1 \leq q < \infty$, and $p \neq q$, $L_{p,q}[0,1]$ and $L_{p,q}[0,\infty)$ are not isomorphic.*

(ii) *For $1 < p \neq q < 2$, $L_{p,q}[0,\infty)$ is not isomorphic to any r.i. function space on $[0,1]$.*

PROOF. (i) For $1 < p < \infty$, $1 \leq q < \infty$, it follows from Lemma 2.1 and [**12**, Lemma 8.10] that the only spaces with symmetric basis which are isomorphic to complemented subspaces of $L_{p,q}[0,1]$ are l_q and l_2. This is not the case in $L_{p,q}[0,\infty)$ since $L_{p,q}[0,\infty)$ obviously contains a complemented copy of $l_{p,q}$ on disjoint vectors.

(ii) This is a simple variation on an argument found in [**12**, Proposition 8.14]. First note that, for $1 < p \neq q < 2$, $L_{p,q}[0,\infty)$ is s-concave for $\max(p,q) < s < 2$. Thus by [**12**, Theorem 5.6], if an r.i. function space X on $[0,1]$ were isomorphic to $L_{p,q}[0,\infty)$, then, up to an equivalent norm, either $X = L_2[0,1]$ or $X = L_{p,q}[0,1]$, but neither option is possible. ■

REMARK. Of course, $L_p[0,1]$ and $L_p[0,\infty)$ are isomorphic. It would be interesting to know whether $L_{p,\infty}[0,1]$ and $L_{p,\infty}[0,\infty)$ are isomorphic. However, since $L_{p,\infty}$ contains a sublattice isomorphic to l_∞ (i.e., Lemma 2.1 also holds for $q = \infty$), $L_{p,\infty}$ will be of only passing interest here.

Our next proposition will show that the closed linear span of a disjointly supported sequence in $L_{p,q}$ contains a complemented copy of l_q.

PROPOSITION 2.3. *Let X be an r.i. function space on $[0,\infty)$ which is q-concave for some $q < \infty$ and which for some $p > 0$ satisfies $\|\chi_{[0,t]}\|_X \geq C^{-1}t^{1/p}$ for all $t > 0$.*

If (f_n) is a disjointly supported sequence of norm-one vectors in X, then there exists a subsequence (f'_n) of (f_n) and an increasing sequence of integers (n_k) such that the normalized blocks

$$F_k = \sum_{i=n_k+1}^{n_k+1} f'_i \Big/ \left\| \sum_{i=n_k+1}^{n_k+1} f'_i \right\|_X .$$

satisfy $F_k^* \to 0$ a.e. as $k \to \infty$

PROOF. By Helly's selection theorem [**22**, p. 221] we may suppose, by passing to a subsequence if necessary, that the sequence (f_n^*) converges a.e. to some decreasing function $f \geq 0$. Moreover, since X is q-concave, we also have $f \in X$ and $\|f\|_X \leq 1$ [**20**, p. 30]. If $f = 0$ we are, or course, done; thus we suppose that $|\{f > 2\delta\}| > 2\delta$ for some fixed $\delta > 0$.

Thus we may rewrite f_n as $g_n + h_n$, where (g_n) is a disjointly supported sequence in X with $g_n^* = f$ for all n, and where (h_n) is a disjointly supported sequence in X with $h_n^* \to 0$ a.e. $(n \to \infty)$. We may further suppose that $|\{|h_n| > \varepsilon_n\}| < \varepsilon_n$ where $\varepsilon_n \downarrow 0$ and $\sum_{n=1}^\infty \varepsilon_n < \delta$. It is easy to see that we also have $|\{|f_n| > \delta\}| > \delta$ for all n.

In order to see the inductive step in our procedure, we will show that for any given $\varepsilon > 0$ and $k \geq 1$ there are integers $n \geq k$ and $N \geq 1$ so that if $F = \sum_{i=n+1}^{n+N} f_i$, then $|\{|F| \geq 2\varepsilon \|F\|_X\}| \leq 2\varepsilon$.

First, take $n \geq k$ so that $\sum_{i=n+1}^\infty \varepsilon_i < \varepsilon$. Next, since X is q-concave, we may choose $N \geq 1$ sufficiently large so that

$$\|f^* \chi_{[0,\varepsilon/N)}\|_X \leq C^{-2}(\delta\varepsilon)^{1+1/p}.$$

But notice that we then also have

$$\begin{aligned} f^*(\varepsilon/N) &\leq \|\chi_{[0,\varepsilon/N)}\|_X^{-1} \cdot \|f^*\chi_{[0,\varepsilon/N)}\|_X \\ &\leq C \cdot N^{1/p} \cdot \varepsilon^{-1/p} \cdot \|f^*\chi_{[0,\varepsilon/N)}\|_X \leq C^{-1} \cdot \delta \cdot (N\delta)^{1/p} \cdot \varepsilon. \end{aligned}$$

Now set $F = \sum_{i=n+1}^{n+N} f_i$ and notice that since $|\{|f_i| > \delta\}| > \delta$ we have

$$\|F\|_X \geq \delta \|\chi_{[0,N\delta)}\|_X \geq C^{-1} \cdot \delta \cdot (N\delta)^{1/p}.$$

Thus,

$$
\begin{aligned}
|\{|F| \geq 2\varepsilon\|F\|_X\}| &= \sum_{i=n+1}^{n+N} |\{|f_i| \geq 2\varepsilon\|F\|_X\}| \\
&\leq \sum_{i=n+1}^{n+N} |\{|g_i| \geq \varepsilon\|F\|_X\}| + \sum_{i=n+1}^{n+N} |\{|h_i| \geq \varepsilon\|F\|_X\}| \\
&\leq N|\{f \geq f^*(\varepsilon/N)\}| + \varepsilon \\
&\leq 2\varepsilon. \quad \blacksquare
\end{aligned}
$$

COROLLARY 2.4. *Let $1 < p < \infty$, $1 \leq q < \infty$, and let (f_n) be a disjointly supported norm-one sequence in $L_{p,q}[0,\infty)$. Then $[f_n]$ contains a complemented copy of l_q.*

REMARK. It has been conjectured that Proposition 2.3 should actually hold without the assumption that $\|\chi_{[0,t]}\|_X \geq C^{-1}t^{1/p}$ for some $p > 0$. However, the following example (suggested to us by W. B. Johnson) shows that some additional assumption is, in fact, necessary. Let $1 < r < 2$ and let X be the space $L_r[0,\infty) + L_2[0,\infty)$ (then $\|f\|_X \sim \|f^*\chi_{[0,1]}\|_r + \|f^*\chi_{[1,\infty)}\|_2$). Now fix $r < s < 2$ and let (f_n) be a disjointly supported sequence in X with $f_n^*(t) = t^{-1/s}$, $0 < t \leq 1$, and $f_n^*(t) = 0$, $t > 1$, for all n. If $F_n = \sum_{i=1}^n f_i$, then a simple computation shows that $\|F_n\|_X \leq C \cdot n^{1/s}$. But then $F_n^*(t) = n^{1/s}t^{-1/s} \geq C^{-1} \cdot t^{-1/s} \cdot \|F_n\|_X$ for $0 < t \leq n$. That is, the conclusion of Proposition 2.3 cannot hold for the sequence (f_n).

THEOREM 2.5. *Let $1 < p < \infty$, $1 \leq q < \infty$, and let X be a subspace of $L_{p,q}[0,\infty)$. Then either X is isomorphic to a strongly embedded subspace of $L_{p,q}[0,1]$, or X contains a complemented copy of l_q.*

Consequently, if $p > 2$, then either X is isomorphic to l_2 (and complemented), or X contains a complemented copy of l_q.

PROOF. If for some $s < \infty$ the restriction map $f \mapsto f\chi_{[0,s]}$ is an isomorphism on X, then we may suppose that X is actually a subspace of $L_{p,q}[0,s]$ and, hence, apply a standard Kadec-Pełczyński argument [**9**, Theorem 4.1 or **20**, Proposition 1.c.8] to conclude that either X is strongly embedded in $L_{p,q}[0,s]$ (and thus also in $L_{p,q}[0,1]$ by way of dilation), or that X contains an unconditional basic sequence equivalent to a disjointly supported norm one sequence (f_n) in $L_{p,q}[0,s]$. Clearly, $f_n^* \to 0$.

If, on the other hand, restriction to $[0,s]$ fails to be an isomorphism on X for every $s < \infty$, then it is easy to find norm-one vectors (x_n) in X and disjointly supported norm-one vectors (f_n) in $L_{p,q}[0,\infty)$ satisfying $\|x_n - f_n\|_{p,q} \to 0$. Corollary 2.4 (and a standard perturbation argument) now finishes the proof. $\blacksquare$

REMARK. Up to this point we have stated all of our observations in the range $1 < p < \infty$, $1 \leq q < \infty$, but only in order to take most immediate advantage of several known results which are implicitly stated in this range. It is not hard to see, though, that virtually all of these observations are actually valid in the entire range $0 < p, q < \infty$ (with only one obvious modification: we must avoid the word "complemented" if $p \leq 1$).

In order to complete the proof of the result stated in the abstract we must consider strongly embedded subspaces of $L_{p,q}[0,1]$ for $p < 2$. Perhaps surprising is that the key step in our proof will be to show that l_p does *not* embed in $L_{p,q}[0,\infty)$, $p \neq q$, $p \neq 2$. In order to do so we will first supply a new "disjointification" procedure (which further underlines the natural role of $I = [0,\infty)$ in our arguments).

Given $f_1, \ldots, f_n \in L_{p,q}[0,1]$ we define the *disjoint sum* $\sum_{i=1}^n \bigoplus f_i$ to be any function $f \in L_{p,q}[0,\infty)$ with $d_f = \sum_{i=1}^n d_{f_i}$. For example, we could take

$$f(t) = \sum_{i=1}^n f_i(t-i+1)\chi_{[i-1,i)}(t) \tag{6}$$

for $0 \leq t \leq n$.

PROPOSITION 2.6. *For $0 < p < \infty$, $p \neq 1$, and $0 < q \leq \infty$, there is a constant C such that*

$$\left\| \sum_{i=1}^n |f_i| \right\|_{p,q} \leq C \left\| \sum_{i=1}^n \bigoplus f_i \right\|_{p,q} \quad \text{for } 0 < p < 1, \tag{7}$$

$$\left\| \sum_{i=1}^n \bigoplus f_i \right\|_{p,q} \leq C \left\| \sum_{i=1}^n |f_i| \right\|_{p,q} \quad \text{for } 1 < p < \infty, \tag{8}$$

for any $f_1, \ldots, f_n \in L_{p,q}[0,1]$.

PROOF. For $p > 1$ and $q \geq 1$, (8) would follow from a version of [**12**, Lemma 7.1]; but more generally (8) follows from Kalton's "property (d)" [**15**], or from [**20**, Proposition 2.a.8]. That is, for $p > 1$, $L_{p,q}$ satisfies $\|f\|_{p,q} \leq C\|g\|_{p,q}$ whenever $f^{**} \leq g^{**}$ (recall that $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds$), and in our setting it is easy to check that $(\sum_{i=1}^n \bigoplus f_i)^{**} \leq (\sum_{i=1}^n |f_i|)^{**}$.

In case $0 < p < 1$ we prove (7) by interpolation. Fix n and define a quasi-linear map $T : L_0[0,n] \to L_0[0,1]$ by $Tf = \sum_{i=1}^n |f_i|$, where $f_i(t) = f(t+i-1)$, $0 \leq t \leq 1$. That is, if f is defined as in (6), then $T(\sum_{i=1}^n \bigoplus f_i) = \sum_{i=1}^n |f_i|$. In order to prove (7) we need only show that T is bounded on $L_r[0,n]$, $0 < r < 1$ (independent of n). But for $r < 1$,

$$\begin{aligned} \|Tf\|_r = \left\| \sum_{i=1}^n |f_i| \right\|_r &\leq \left(\sum_{i=1}^n \|f_i\|_r^r \right)^{1/r} \\ &= \left\| \sum_{i=1}^n \bigoplus f_i \right\|_r = \|f\|_r. \quad \blacksquare \end{aligned}$$

As a corollary, we get an improvement on a result from [**4**].

COROLLARY 2.7. *For $0 < p < \infty$, $p \neq 2$, and $0 < q \leq \infty$, there is a constant C such that*

$$\left\| \left(\sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_{p,q} \leq C \left\| \sum_{i=1}^n \bigoplus f_i \right\|_{p,q} \quad \text{for } 0 < p < 2, \tag{9}$$

$$\left\| \sum_{i=1}^n \bigoplus f_i \right\|_{p,q} \leq C \left\| \left(\sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_{p,q} \quad \text{for } 2 < p < \infty, \tag{10}$$

for any $f_1, \ldots, f_n \in L_{p,q}[0,1]$.

REMARK. It is shown in [4] that neither (7) nor (8) can hold when $p = 1$, $q \neq 1$ (i.e., neither (9) nor (10) can hold when $p = 2$, $q \neq 2$). But notice that Corollary 2.7 easily holds (by dilation, or by modifying (6)) for $f_1, \dots, f_n \in L_{p,q}[0, s]$, $s < \infty$ (and $p \neq 2$).

THEOREM 2.8. *Let $0 < p < q < \infty$, $p \neq q$, $p \neq 2$. Then l_p does not embed into $L_{p,q}[0, \infty)$.*

PROOF. By Theorem 2.5 we may suppose that $p < 2$. We will first consider the case $p < q$. Suppose that $(f_i) \subset L_{p,q}[0, \infty)$ is a normalized sequence K-equivalent to the unit vector basis in l_p (we may also suppose that each f_i has compact support). Next let (g_i) be a disjointly supported sequence in $L_{p,q}[0, \infty)$ with $d_{g_i} = d_{f_i}$ for all i. Now the Maurey-Khinchine inequality [20, Theorem 1.d.6] and Corollary 2.7(9) yield

$$C^{-1}K^{-1}\left\|\sum_{i=1}^{n} a_i f_i\right\|_{p,q} \leq \left\|\left(\sum_{i=1}^{n} |a_i f_i|^2\right)^{1/2}\right\|_{p,q} \leq C\left\|\sum_{i=1}^{n} a_i g_i\right\|_{p,q}$$

for any n and scalars (a_i). But since $L_{p,q}$ satisfies an upper p-estimate, we would then have

$$\begin{aligned} K^{-1}\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} &\leq \left\|\sum_{i=1}^{n} a_i f_i\right\|_{p,q} \leq C'K\left\|\sum_{i=1}^{n} a_i g_i\right\|_{p,q} \\ &\leq C''K\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}. \end{aligned}$$

That is, (g_i) would also be equivalent to the l_p basis and, again, this is impossibly by Corollary 2.4.

We now consider the case $q < p$. First notice that if X is a subspace of $L_{p,q}[0, \infty)$ isomorphic to l_p, then by Theorem 2.5 we must have that X strongly embeds in $L_{p,q}[0, 1]$. But a strongly embedded subspace of $L_{p,q}[0, 1]$ would also embed (strongly) in $L_{p,r}[0, 1]$ for any $p < r < \infty$, and this is impossible for $X \sim l_p$ by the first part of the proof. ■

REMARK. The proof of Theorem 2.8 shows that l_p does not strongly embed in $L_p[0, 1]$ for $0 < p < 2$. For $1 \leq p < 2$ this result is due to Rosenthal [25] (by a very different method; see also [8]), and for $0 < p < 1$ is implicit in Kalton's result [14] that l_p does not embed in $L_{p,q}[0, 1]$ for $0 < p < 1$, $p < q < \infty$ (but Kalton's proof appears to require $p < 1$). Kalton has also shown that $L_{p,\infty}[0, 1]$ contains a complemented copy of $L_p[0, 1]$ for $1 \leq p < 2$ [16].

Finally we may complete the proof of our main result.

COROLLARY 2.9. *Let $1 < p < \infty$, $1 \leq q < \infty$, and let X be a subspace of $L_{p,q}[0, \infty)$. Then either X is isomorphic to a strongly embedded subspace of $L_p[0, 1]$, or X contains a complemented copy of l_q.*

PROOF. Of course we need only show that a strongly embedded subspace of $L_{p,q}[0, 1]$ is actually strongly embedded in $L_p[0, 1]$. This is obvious for $q \leq p$, so suppose X is strongly embedded in $L_{p,q}[0,1]$, $1 < p < 2$, $p < q < \infty$. But then X is strongly embedded in $L_r[0,1]$ for every $r < p$, and by Theorem 2.8 we know that l_p does *not* embed in X. By a result due to Guerre and Levy [**10**], this implies that X does, in fact, strongly embed in $L_p[0,1]$. ∎

REFERENCES

1. Z. Altshuler, P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces*, Israel J. Math. **15** (1973), 144–155.
2. J. Bergh and J. Löfstrom, *Interpolation spaces*, Grundlehren der mathematischen Wissenschaften, vol. 223, Springer-Verlag, Berlin, Heidelberg, and New York, 1976.
3. N. L. Carothers, *Rearrangement invariant subspaces of Lorentz function spaces.* II, Rocky Mountain J. Math. **17** (1987), 607–616.
4. N. L. Carothers and S. J. Dilworth, *Equidistributed random variables in $L_{p,q}$*, J. Funct. Anal. (to appear).
5. P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces.* II, Israel J. Math. **17** (1974), 191–218.
6. J. Creekmore, *Type and cotype in Lorentz $L_{p,q}$ spaces*, Indag. Math. **43** (1981), 145–152.
7. J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math., vol. 92, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1984.
8. P. Enflo and H. P. Rosenthal, *Some results concerning $L_p(\mu)$-spaces*, J. Funct. Anal. **14** (1973), 325–348.
9. T. Figiel, W. B. Johnson and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure with applications to Lorentz function spaces*, J. Approx. Theory **13** (1975), 297–312.
10. S. Guerre and M. Levy, *Espaces l_p dans les sous-espaces de L_1*, Trans. Amer. Math. Soc. **279** (1983), 611–616.
11. R. A. Hunt, *On $L(p,q)$ spaces*, Enseign. Math. **12** (1966), 249–274.
12. W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. **217** (1979).
13. M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1962), 161–176.
14. N. J. Kalton, *Linear operators on L_p for $0 < p < 1$*, Trans. Amer. Math. Soc. **259** (1980), 319–355.
15. ——, *Compact and strictly singular operators on certain function spaces*, Arch. Math. **43** (1984), 66–78.
16. ——, *Banach spaces embedding into L_0*, Israel J. Math. **52** (1985), 305–319.
17. M. Levy, Thesis, Université Paris VI, 1980 (= C.R. Acad. Sci. Paris **289** (1979), 675–677).
18. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequences spaces.* II, Israel J. Math. **11** (1970), 355–379.
19. ——, *Classical Borel Spaces.* I. *Sequence Spaces*, Ergebnisse Math. Grenzgebiete, Bd. 92, Springer-Verlag, Berlin, Heidelberg, and New York, 1977.
20. ——, *Classical Banach Spaces.* II. *Function Spaces*, Ergebnisse Math. Grenzgebiete, Bd. 97, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
21. J. L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. **19** (1964), 5–68.
22. I. P. Natanson, *Theory of functions of a real variable*, Vol. 1, Ungar, New York, 1961.
23. R. O'Neil, *Integral transforms and tensor products on Orlicz spaces and $L(p,q)$ spaces*, J. Analyse Math. **21** (1968), 1–276.
24. Y. Raynaud, Thesis, Université Paris VII, 1981.
25. H. P. Rosenthal, *On subspaces of L_p*, Ann. of Math. **97** (1973), 344–373.
26. E. M. Stein and G. Weiss, *Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

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