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OSCILLATORY SOLUTIONS FOR CERTAIN DELAY-DIFFERENTIAL EQUATIONS

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ABSTRACT. The existence of oscillatory solutions for a certain class of scalar first order delay-differential equations is proved. An application to a delay logistic equation arising in certain models for population variation of a single specie in a constant environment with limited resources for growth is considered.

It is known (cf. [**1**, **2**]) that all solutions of the delay logistic equations

$$N'(t) = N(t)(a - bN(t) - N(t-1)), \qquad t > 0, \tag{1}$$

with $N(t) = N_0(t) > 0$, $-1 \leq t \leq 0$, N_0 continuous, and a and b positive constants, satisfy $N(t) \to a/(b+1)$ as $t \to \infty$ whenever $b > 1$. In [**3**] it was shown that for any $b > 0$, there exists $a(b) > 0$ and that if $0 < a < a(b)$, there exist solutions $N(t)$ of (1) which do not oscillate about the equilibrium $N = a/(b+1)$; in particular, such that, $N(t) > a/(b+1)$ for $t \geq 0$. It is the purpose of this paper to show that for this same $a(b)$, if $a > a(b)$, there exist oscillatory solutions about this equilibrium solution. In case $b < 1$, this is known; in fact, a Hopf bifurcation (cf. [**1**]) shows the existence for certain a of nonconstant positive periodic solutions. However, if $b > 1$, the fact that some solutions of (1) approach $a/(b+1)$ in an oscillatory fashion seems to be new.

The above mentioned result for (1) will follow from a result for a more general scalar delay-differential equation of the form

$$y'(t) = L(y_t) + N(t, y_t), \qquad t > 0. \tag{2}$$

Here $y_t = y(t+\theta)$, $-1 \leq \theta \leq 0$, and we assume

(H_1) $L(\phi)$ is continuous and linear on $C = C([-1,0], R)$ and $N(t,\phi)$ is continuous on $R \times C$ and satisfies

$$|N(t,\phi)| \leq M(t)||\phi||^2, \qquad \phi \in C,\ ||\phi|| \leq B_0,\ t \geq 0;$$

where the norm in C is defined by $||\phi|| = \sup\{|\phi(\theta)|\colon\ -1 \leq \theta \leq 0\}$, and $\int_0^\infty M(t)\,dt < \infty$;

(H_2) The characteristic equation for

$$y'(t) = L(y_t) \tag{3}$$

has a pair of simple pure imaginary roots $\pm i\beta$, $\beta > 0$, and all other roots have negative real parts.

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REMARK 1. Under assumption (H_2), there exists a nonconstant periodic solution $y^*(t)$ of (3) and positive numbers ρ^* and B, $\rho^* < 1$, $B < B_0/2$, such that

$$\max\{y^*(t) : t \in R\} \geq \rho^*, \ \min\{y^*(t) : t \in R\} \leq -\rho^*,$$
$$|y^*(t)| \leq B, \qquad t \in R.$$

This follows from standard theory for solutions of (3); cf., for example, Hale's monograph [**4**].

DEFINITION. The real-valued function $f(t)$ on $[0, \infty)$ is oscillatory if there exist $t_n \to \infty$ as $n \to \infty$, $t_{n+1} > t_n$, such that $(-1)^n f(t_n) > 0$, $n = 1, 2, \ldots$.

REMARK 2. If $f(t)$ is continuous and oscillatory in this sense, clearly f must have an unbounded sequence of zeros and cannot be identically zero on any half infinite interval $[t_0, \infty)$, $t_0 \geq 0$.

THEOREM 1. *If* (H_1) *and* (H_2) *hold, there exists* $\delta_0 > 0$ *such that for each* δ, $0 < \delta < \delta_0$, (2) *has an oscillatory solution* $y = w(t)$ *such that* $|w(t)| \leq \delta$, $t \geq 0$.

PROOF. Let $u(t)$ be the fundamental solution for (3); i.e., let $u(t)$ solve (cf. appendix)

$$\begin{aligned} &u'(t) = L(u_t), \qquad t > 0, \\ &u(0) = 1, \\ &u(t) = 0, \qquad -1 \leq t < 0. \end{aligned} \tag{4}$$

From (H_2) it follows that there exists $K > 0$ such that $|u(t)| \leq K$, $t \geq 0$; again cf. [**4**, Chapter 7]. For this K, and ρ^* and B as in Remark 1, fix $\varepsilon > 0$ such that

$$\varepsilon \int_0^\infty M(t)\, dt \leq \frac{\rho^*}{8B^2K} < (4BK)^{-1}; \tag{5}$$

note that $\rho^* \leq B$.

Let $X(B)$ denote the set of real functions z continuous on $[-1, \infty)$ such that $z(t) = y^*(t)$, $-1 \leq t \leq 0$, where $y^*(t)$ is the periodic solution of (3) as described in Remark 1, and $|z(t)| \leq 2B$, $t \geq 0$. With the topology of uniform convergence on compact subsets of $[-1, \infty)$, the set X of all real functions continuous on $[-1, \infty)$ is a locally convex linear topological space over the reals, and clearly $X(B) \subset X$. Define the map T on $X(B)$ to X by

$$\begin{aligned} (Tz)(t) &= y^*(t) + \frac{1}{\varepsilon} \int_0^t u(t-s) N(s, \varepsilon z_s)\, ds, \qquad t > 0, \\ &= y^*(t), \qquad -1 \leq t \leq 0, \end{aligned} \tag{6}$$

for any $z \in X(B)$.

Using (5) with (H_1) and the boundedness property of $u(t)$, we have

$$(Tz)(t) \leq B + 4KB^2\varepsilon \int_0^t M(s)\, ds \leq 2B, \qquad t \geq 0;$$

therefore $Tz \in X(B)$.

Using (6) and the properties of $u(t)$ given in (4) it follows that

$$\begin{aligned} \frac{d}{dt}(Tz)(t) &= y^{*\prime}(t) + \frac{1}{\varepsilon} N(t, \varepsilon z(t)) + \frac{1}{\varepsilon} \int_0^t u'(t-s) N(s, \varepsilon z(s))\, ds \\ &= y^{*\prime}(t) + \frac{1}{\varepsilon} N(t, \varepsilon z(t)) + \frac{1}{\varepsilon} \int_0^t L(u_{t-s}) N(s, \varepsilon z(s))\, ds. \end{aligned}$$

So since L is bounded, $z \in X(B)$, $u(t)$ is bounded for $t > 0$, and N satisfies the conditions in (H_1), it follows that there exists a constant $C(\varepsilon)$ and that

$$\frac{d}{dt}(Tz)(t) \le C(\varepsilon), \qquad t > 0.$$

By a standard argument using the Ascoli-Arzela theorem, it then follows that $TX(B)$ is precompact in the topology of X, and by the Schauder-Tychonov fixed point theorem, there exists a $z^* \in X(B)$ such that

$$\begin{aligned} z^*(t) &= y^*(t) + \frac{1}{\varepsilon}\int_0^t u(t-s)N(s, \varepsilon z^*(s))\,ds, \qquad t > 0, \\ z^*(t) &= y^*(t), \qquad -1 \le t \le 0. \end{aligned} \tag{7}$$

Since $u(t)$ is a fundamental solution for (3), it follows that $z^*(t)$ solves

$$\begin{aligned} z'(t) &= L(z_t) + \frac{1}{\varepsilon}N(t, \varepsilon z_t), \qquad t > 0, \\ z(t) &= y^*(t), \qquad -1 \le t \le 0, \end{aligned}$$

and so $y(t) = \varepsilon z^*(t)$ solves (2) for $t > 0$ with $y(t) = \varepsilon y^*(t)$ for $-1 \le t \le 0$.

If

$$R_0(t) = \frac{1}{\varepsilon}\int_0^t u(t-s)N(s, \varepsilon z^*(s))\,ds, \qquad t \ge 0,$$

then $z^*(t) = y^*(t) + R_0(t)$, $t \ge 0$, and using the properties of u and N and the fact that $z^* \in X(B)$ it follows that

$$|R_0(t)| \le 4B^2K\varepsilon \int_0^t M(s)\,ds \le \frac{\rho^*}{2}, \qquad t \ge 0.$$

So

$$|z^*(t) - y^*(t)| \le \rho^*/2, \qquad t \ge 0. \tag{8}$$

But using the properties of $y^*(t)$ mentioned in Remark 1, there exists $t_n \to \infty$ as $n \to \infty$, $t_{n+1} > t_n$, such that $y^*(t_n) \ge \rho^*$, $n = 1, 2, \ldots$. Using (8) it follows easily that

$$z^*(t_n) \ge \rho^*/2, \qquad n = 1, 2, \ldots.$$

Similarly, there exists a sequence $\tau_n \to \infty$ as $n \to \infty$, $\tau_{n+1} > \tau_n$, such that $y^*(\tau_n) \le -\rho^*$, and so

$$z^*(\tau_n) \le -\rho^*/2, \qquad n = 1, 2, \ldots.$$

Thus the solution $y(t) = \varepsilon z^*(t) \equiv w(t)$ of (2) is oscillatory. Now define ε_0 to be the supremum of the set of all $\varepsilon > 0$ for which this argument holds. Since for such $\varepsilon > 0$, $|w(t)| \le \varepsilon B$, with B as in Remark 1, and if we take $\delta_0 = 2\varepsilon_0 B$, our theorem is proved. Note that from (5), $\varepsilon_0 \le (4BKM)^{-1}$, where $M = \int_0^\infty M(t)\,dt$.

REMARK 3. If β is as in (H_2), it can be shown that the t_n and τ_n in our proof above can be chosen such that

$$t_{n+1} - t_n \le 2\pi/\beta, \quad \text{and} \quad \tau_{n+1} - \tau_n \le 2\pi/\beta.$$

This follows because $y^*(t)$ can be chosen to be a linear combination of $\sin \beta t$ and $\cos \beta t$. We omit the details.

We now return to the delay logistic equation (1) with $b > 1$. If we make the change of variables $x(t) = N(t) - a/(b+1)$ (1) becomes

$$x'(t) = -(a/(b+1) + x(t))(bx(t) + x(t-1)), \tag{9}$$

and the linear part of (9) is the equation

$$x'(t) = -(a/(b+1))(bx(t) + x(t-1)). \tag{10}$$

It is not difficult to see that all roots of the characteristic equation for (10) have negative real part, cf. [**5**]. From a result in [**3**], it also follows that if

$$a > (b+1)/m(b), \tag{11}$$

where $m(b)$ is the unique root of $b = m\ (\log m - 1)$, then all roots of this characteristic equation are nonreal. A direct examination of this characteristic equation also shows that all nonreal roots must be simple.

Under the change of variable $y(t) = x(t)\exp(\mu t)$, where μ is a real constant, (9) becomes

$$y'(t) = A(\mu)y(t) + B(\mu)y(t-1) + f(y(t), y(t-1))\exp(-\mu t) \tag{12}$$

where $A(\mu) = \mu - ab/(b+1)$, $B(\mu) = -ae^{\mu}/(b+1)$, and

$$f(y,z) = -(by^2 + yze^{\mu}).$$

It is easy to see that if α is the real part of a root of the characteristic equation for (9), then $\mu + \alpha$ is the real part of a corresponding root of the characteristic equation for the linear part of (12), namely

$$y'(t) = A(\mu)y(t) + B(\mu)y(t-1). \tag{13}$$

So if we choose $\mu = -\max\{\operatorname{Re}\lambda\colon \lambda \text{ is a root of the characteristic equation for } (10)\}$, then the characteristic equation for (13) has pure imaginary roots $\pm i\beta$, $\beta > 0$, which are simple if (11) holds. Also all other roots of this equation for (13) have negative real parts. Clearly $\mu > 0$. So we see that all the hypotheses of Theorem 1 are satisfied for (11) and we have the following.

THEOREM 2. *If $b > 1$ and $a > (b+1)/m(b)$, where $m(b)$ is as defined above, then there exist oscillatory solutions of* (9) *of arbitrarily small amplitude*; *i.e., there exist solutions of* (1) *which oscillate about $a/(b+1)$.*

The proof of this theorem now follows easily, since by Theorem 1, there exist such oscillatory solutions $y(t)$ of (12) and so the corresponding solutions $x(t) = y(t)\exp(-\mu t)$ are also oscillatory.

An open question presents itself: under the hypotheses of Theorem 2, are all solutions of (9) oscillatory?

APPENDIX. In the strict sense, the initial function on $[-1,0]$ for the equation defining $u(t)$ in (4) is not in C. What is really involved here (a point not entirely clear in [**4**]) is that $u(t)$ solves the initial value problem

$$\begin{aligned} u'(t) &= \int_{-t}^{0} u(t+s)\,d\eta(s), \qquad 0 \le t < 1, \\ &= \int_{-1}^{0} u(t+s)\,d\eta(s), \qquad t \ge 1, \\ u(0) &= 1, \end{aligned}$$

where $\eta(s)$ is a function of bounded variation which by the Riesz representation theorem characterizes L; i.e. is such that $L(\phi) = \int_{-1}^{0} \phi(s)\, d\eta(s)$ for $\phi \in C$. This initial value problem can be shown to have a solution in a fairly standard way such as by the method of successive approximations.

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