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# STABLE MAPS INTO THE HILBERT CUBE

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ABSTRACT. A map into the Hilbert cube is stable if each composition with projection onto a finite number of factors is stable. We prove that a map from a compact metric space into the Hilbert cube is stable if and only if it is universal. As a consequence, the composition of a stable map with any self homeomorphism of the Hilbert cube is also stable.

**1. Introduction.** All spaces are separable metric spaces. A map $f\colon X \to I^n$ is said to be *stable* if there does not exist a map $g\colon X \to S^{n-1}$ with $f|_{f^{-1}(S^{n-1})} = g|_{f^{-1}(S^{n-1})}$. Stable maps are also known as Alexander-Hopf essential maps [**N**, **GT**]. Krasinkiewicz has given a general definition of essential maps into the product of manifolds [**K**] that coincides with the definition of stable maps in the cases under consideration. It is well known that a space has covering dimension greater than or equal to $n$ if and only if it admits a stable map into $I^n$ [**HW**]. Note that if $f\colon X \to I^n$ is a stable map and $h$ is any self-homeomorphism of $I^n$, then the composition $h \circ f$ is also stable, since $h(S^{n-1}) = S^{n-1}$.

Let $I^\infty = \prod_{i=1}^{\infty}[-1,1]_i$ denote the Hilbert cube, and for each $n$ let $p_n\colon I^\infty \to I^n$ be the projection map onto the first $n$ factors. The subsets $A_n = \{(x_i) \in I^\infty | x_n = -1\}$ and $B_n = \{(x_i) \in I^\infty | x_n = 1\}$ are referred to as *faces* of the Hilbert cube. A map $f\colon X \to I^\infty$ from a compact metric space into the Hilbert cube is said to be *stable* if the composition $p_n \circ f\colon X \to I^n$ is stable for each $n$. See [**W** and **B**] for a more detailed description of stable maps. In particular, Walsh showed that a map $f\colon X \to I^\infty$ from a compact metric space is stable if and only if the collection of pairs $\{(f^{-1}(A_i), f^{-1}(B_i)) | i = 1, 2, \dots\}$ is an essential family for $X$, i.e., for any sequence of separators $\{S_i\}$ of $f^{-1}(A_i)$ and $f^{-1}(B_i)$, $\bigcap_{i=1}^{\infty} S_i \neq \varnothing$. It follows that a compact metric space admits a stable map into the Hilbert cube if and only if it is strongly infinite dimensional.

Our goal is to prove a result for stable maps into the Hilbert cube which is analogous to the result noted above for stable maps into $n$-cells. Namely, we show that if $f\colon X \to I^\infty$ is stable and $h$ is any self-homeomorphism of $I^\infty$, then the composition $h \circ f$ is also stable. This gives a partial answer to a question of J. Krasinkiewicz [**K**, Problem 1]. What we need for the proof is a characterization of stable maps in terms of a property preserved by self-homeomorphisms. *Universality*, a concept introduced by Holsztyński [**H1**] and widely used in the study of fixed point theory (see [**H2**, **H3**, **H4**, **H5**, **H6**, **H7**, **GT** and **N**]), is such a property. A map

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$f\colon X \to Y$ is said to be *universal* if for every map $g\colon X \to Y$ there exists a point $p$ in $X$ with $f(p) = g(p)$. We will show that a map from a compact metric space into the Hilbert cube is stable if and only if it is universal. Then the desired result on preservation of stability by compositions with self-homeomorphisms of $I^\infty$ is an immediate corollary.

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**2. Universal maps and stable maps.** The following result is contained in [**GT**, **N**] and implicitly in [**H1**]. For completeness, we include a short proof.

THEOREM 1. *Let $n$ be in $\mathbf{Z}_+$. A map $f\colon X \to I^n$ is stable if and only if it is universal.*

PROOF. Suppose that $f\colon X \to I^n$ is a stable map. If $f$ were not universal, then we could find a map $g\colon X \to I^n$ so that $g(p) \neq f(p)$ for every point $p$ in $X$. Consider $S^{n-1}$ as the boundary of $I^n$ in the usual manner, and define $h\colon X \to S^{n-1}$ by setting $h(p)$ equal to the intersection of the ray containing $f(p)$ which emanates from $g(p)$ and $S^{n-1}$. Clearly $h$ is continuous and agrees with $f$ on $f^{-1}(S^{n-1})$, contradicting the stability of $f$. Therefore, $f$ must be universal.

For the converse, suppose that $f\colon X \to I^n$ is universal and again consider $S^{n-1}$ as the boundary of $I^n$. If $f$ were not stable, then we could find a map $g\colon X \to S^{n-1}$ which agrees with $f$ on $f^{-1}(S^{n-1})$. Composing with the antipodal map $\alpha\colon S^{n-1} \to S^{n-1}$ would then give a map $\alpha \circ g\colon X \to S^{n-1} \subset I^n$ so that $\alpha \circ g(p) \neq f(p)$ for any point $p$ in $X$, contradicting the universality of $f$. Thus, $f$ must be stable. $\square$

The next result is the tool needed to link stability and universality of maps of compacta into the Hilbert cube.

THEOREM 2. *A map $f\colon X \to I^\infty$ from a compact metric space into the Hilbert cube is universal if and only if each composition $p_n \circ f\colon X \to I^n$ is universal.*

PROOF. Assume that $f$ is universal. Fix a positive integer $n$. Consider the Hilbert cube as $I^n \times \prod_{j>n}[-1,1]_j$, and let a map $g\colon X \to I^n$ be given. By choosing a point $y_j$ in $[-1,1]_j$ for each $j > n$, we may assume that $g$ is a map into the Hilbert cube. Thus, since $f$ is universal, there exist a point $p$ in $X$ so that $g(p) \times \prod_{j>n}\{y_j\} = f(p)$. Thus, $g(p) = p_n \circ f(p)$, and $p_n \circ f$ is shown to be universal.

For the converse, we assume that the Hilbert cube has the metric given by $d(y,y') = \sum_{i=1}^\infty (|y_i - y_i'|/2^i)$. Suppose that $p_n \circ f$ is universal for each positive integer $n$. If $f$ were not universal, then there exists a map $g\colon X \to I^\infty$ so that for every point $p$ in $X$, $g(p) \neq f(p)$. By the compactness of $X$, there exists a number $\delta > 0$ so that for every point $p$ in $X$ the distance $d(g(p), f(p)) > \delta$. But we can choose a positive integer $N$ so that the $\operatorname{diam}(\prod_{i=N+1}^\infty [-1,1]_i) < \delta$. This would imply that, for any point $p$ in $X$, $p_N \circ g(p) \neq p_N \circ f(p)$ contradicting the universality of $p_N \circ f(p)$. Therefore, $f$ must be universal. $\square$

We are now ready to obtain the results mentioned at the end of the previous section.

COROLLARY 1. *A map $f\colon X \to I^\infty$ from a compact metric space into the Hilbert cube is stable if and only if it is universal.*

PROOF. This is immediate from Theorems 1 and 2. $\square$

COROLLARY 2. *If $f: X \to I^\infty$ is stable and $h$ is a self-homeomorphism of $I^\infty$, then $h \circ f$ is also stable.*

PROOF. Clearly, universality of maps is preserved by composition with self-homeomorphisms of $I^\infty$. By Corollary 1, the same is true for stable maps. □

COROLLARY 3. *Let $\{(C_i, D_i)\}_{i=1}^\infty$ be an essential family for a strongly infinite-dimensional compact space $X$, and let $f$ be a map from $X$ into $I^\infty$ so that $C_i = f^{-1}(A_i)$ and so that $D_i = f^{-1}(B_i)$ for each $i$. If $h$ is any self-homeomorphism of $I^\infty$, then $\{((h \circ f)^{-1}(A_i), (h \circ f)^{-1}(B_i))\}_{i=1}^\infty$ is also an essential family for $X$.*

PROOF. This follows immediately from Corollary 2 and Walsh's characterization of stability mentioned in §1. □

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