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A CONVERSE TO A RESIDUAL FINITENESS THEOREM OF G. BAUMSLAG

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ABSTRACT. It is shown that if at least two of the factor groups of a nontrivial amalgamated free product G satisfy nontrivial identities, then a special form of the profinite closure of the associated subgroups is necessary (as well as sufficient) for the residual finiteness of G. An example shows that the necessity no longer holds if only one of the factor groups satisfies an identity.

A group G is said to be residually finite if the intersection of all normal subgroups of finite index in G is trivial. In [**1**] G. Baumslag gives a sufficient condition for an amalgamated free product of two groups to be residually finite. The condition can be stated for an arbitrary number of factors as follows: let

$$G = \langle *A_i \colon H_i = H_j \text{ via } \theta_{ij} \text{ for all } i,j \in \Lambda\rangle \tag{1}$$

be a proper amalgamated free product, so each H_i is a proper subgroup of A_i, and the $\theta_{ij}\colon H_i \to H_j$ are isomorphisms satisfying the consistency conditions $\theta_{ij}\theta_{jk} = \theta_{ik}$ and $\theta_{ij}\theta_{ji} = \theta_{ii} = \mathrm{id}_{H_i}$. Let I be the set of all sequences $P = (P_i)_{i\in\Lambda}$ such that $P_i \lhd_f A_i$ for all i, $(P_i \cap H_i)\theta_{ij} = P_j \cap H_j$ for all i, j, and there exists an integer m such that $|A_i\colon H_i| \leq m$ for all i. Given $P \in I$ one can form the amalgamated free product G_P of the groups $\bar{A}_i = A_i/P_i$ with amalgamated subgroups $\bar{H}_i$ and isomorphisms induced by the θ_{ij}. There is an obvious epimorphism $\pi_P\colon G \to G_P$, and since each G_P is residually finite it is easy to see that the following holds.

THEOREM 1. *Let G be as in* (1). *Then G is residually finite if and only if* $\bigcap_{P\in I} \ker \pi_P = \langle 1\rangle$.

Baumslag's criterion [**1**, Proposition 3] reads as follows:

Let G as in (1). *Assume that*

$$H_i = \bigcap_{P\in I} H_i P_i \quad \textit{for all } i \in \Lambda, \tag{2}$$

$$\bigcap_{P\in I} P_i = \langle 1\rangle \quad \textit{for all } i \in \Lambda. \tag{3}$$

Then G is residually finite.

This follows from Theorem 1 since (2) and (3) ensure that any nontrivial element of G can be mapped to an element of the same length in some G_P.

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It is trivial that (3) is also necessary for the residual finiteness of G, since $\bigcap_{P\in I} P_i = A_i \cap \bigcap_{P\in I} \ker \pi_P$ for all i. The results on the residual finiteness of amalgamated free products published so far either utilize (2), or have hypotheses which imply (2). This raises the question of the necessity of this condition. In this direction Gregorac [**2**] shows that if two of the A_i satisfy nontrivial identities and either $|A_i\colon H_i| \geq 3$ or the identities do not hold in the infinite dihedral group, then (2) is also necessary for the residual finiteness of G. These restrictions, however, are by no means essential, as the following shows:

THEOREM 2. *Let G be as in* (1), *and assume that at least two of the A_i satisfy nontrivial identities. Then conditions* (2) *and* (3) *are necessary as well as sufficient for the residual finiteness of G.*

That some restriction on the A_i is necessary is shown by the following example:

EXAMPLE. *There exists a residually finite amalgamated free product $G = \langle A*B;\ H = K \text{ via } \theta\rangle$ where $H \subsetneq A$, $K \subsetneq B$, the group A is torsion-free abelian, and H is not profinitely closed in A.*

PROOF OF THEOREM 2. We need to recall the following fact [**4**, Lemma 2]: a group A satisfying a nontrivial identity also satisfies a nontrivial identity of the form

$$(4)\qquad \begin{aligned} W = W(y, x_1, x_2) &= W_0 y^{\varepsilon_1} W_1 y^{\varepsilon_2} \cdots y^{\varepsilon_n} W_n, \\ &\text{where } n \geq 1,\ \varepsilon_r = \pm 1 \quad \text{for all } r, \\ &\text{and}\quad W_0, \ldots, W_n \in \{x_1^{\pm 1}, x_2^{\pm 1},\ (x_1 x_2^{-1})^{\pm 1}\}. \end{aligned}$$

We also write $W(A) = \langle W(a_1, a_2, a_3)\colon a_i \in A\rangle$. Assume that G is residually finite, and suppose $H_k \neq \bigcap_{P\in I} H_k P_k$ for some $k \in \Lambda$. Throughout, h denotes a fixed element in $(\bigcap_{P\in I} H_k P_k)\backslash H_k$. By assumption two of the factor groups, A_i and A_j, say, satisfy nontrivial identities. There are two separate cases to be considered.

Case 1. $k \notin \{i, j\}$. We first show that $|A_i\colon H_i| = 2 = |A_j\colon H_j|$. For assume, to the contrary, that $|A_i\colon H_i| \geq 3$, and let $\{1, a_1, a_2\}$ be part of a right transversal of H_i in A_i. We may suppose that A_i satisfies an identity W as in (4). Consider the element

$$g = W(h, a_1, a_2) = W_0(a_1, a_2) h^{\varepsilon_1} \cdots h^{\varepsilon_n} W_n(a_1, a_2).$$

Clearly g is reduced as written since each $W_r(a_1, a_2) \in A_i\backslash H_i$. Hence $g \neq 1$. If $P \in I$ then $h\pi_P \in H_k\pi_P = H_i\pi_P \subseteq A_i\pi_P$, whence $g\pi_P \in W(A_i)\pi_P = \langle 1\rangle$ since A_i satisfies W. We have therefore found a nontrivial element in $\bigcap_{P\in I} \ker \pi_P$, contradicting the residual finiteness of G, by Theorem 1. Since $A_i \neq H_i$ it follows that we must have $|A_i\colon H_i| = 2$. Similarly $|A_j\colon H_j| = 2$. From this we obtain another contradiction, showing that Case 1 cannot arise. To this end choose $b_i \in A_i\backslash H_i$ and $b_j \in A_j\backslash H_j$. Now H_k, being isomorphic to $H_i \subseteq A_i$, satisfies an identity W of the form (4). Consider the element

$$g = W(h, h^{b_i}, h^{b_j}) = W_0(h^{b_i}, h^{b_j}) h^{\varepsilon_1} \cdots h^{\varepsilon_n} W_n(h^{b_i}, h^{b_j}).$$

For $0 \leq s \leq n$ the element $W_s(h^{b_i}, h^{b_j})$ is one of $(b_i^{-1} h b_i)^{\pm 1}$ or $(b_j^{-1} h b_j)^{\pm 1}$ or $(b_i^{-1} h b_i b_j^{-1} h^{-1} b_j)^{\pm 1}$. Since b_i, b_j and h come from different factors each W_s is reduced as written and begins and ends with some $b^{\pm 1}$. Thus $g \neq 1$. Choose any $P \in I$. Then $h\pi_P \in H_k\pi_P$, and since H_i is normalized by b_i we have

$$h^{b_i}\pi_P \in H_k^{b_i}\pi_P = H_i^{b_i}\pi_P = H_i\pi_P = H_k\pi_P.$$

Similarly $h^{b_j}\pi_P \in H_k\pi_P$, and consequently $g\pi_P \in W(H_k)\pi_P = \langle 1 \rangle$. This is a contradiction. Case 1, therefore, cannot arise.

Case 2. $k \in \{i, j\}$, say $k = i$. The first part of the proof of Case 1 shows that $|A_j : H_j| = 2$. Let $\{1, a_1, a_2\}$ be part of a right transversal of H_k in A_k. We may assume that $A_k(= A_i)$ satisfies an identity W of the form (4). Choose $b \in A_j \backslash H_j$ (so b normalizes H_j), and let

$$g = W(h^b, a_1, a_2) = W_0(a_1, a_2)b^{-1}h^{\varepsilon_1}b \cdots b^{-1}h^{\varepsilon_n}bW_n(a_1, a_2).$$

By construction $h^{\pm 1}$ and the W_s come from $A_k \backslash H_k$, while $b^{\pm 1} \in A_j \backslash H_j$. Thus $g \neq 1$. For any $P \in I$ we have

$$h^b\pi_P \in H_k^b\pi_P = H_j^b\pi_P = H_j\pi_P = H_k\pi_P \subseteq A_k\pi_P$$

and consequently $g\pi_P \in W(A_k)\pi_P = \langle 1 \rangle$. This is impossible in view of the residual finiteness of G. We have now shown that H_k has index 2 in A_k. Consider the sequence $Q = (Q_\lambda)_{\lambda \in \Lambda}$ where $Q_k = H_k$ and $Q_\lambda = A_\lambda$ if $\lambda \neq k$. Clearly $Q \in I$. But then $\bigcap_{P \in I} H_kP_k \subseteq H_kQ_k = H_k$. This final contradiction proves the result.

We now turn to the example. Let $A = \times_{i=1}^{\infty}\langle a_i \rangle$ be a direct product of infinite cyclic groups, and let $r \geq 2$ be a fixed integer. Consider any set $\{h_i : i = 1, 2, \dots\}$ of elements of A, and let

$$G = \langle A, x_i,\ i = 1, 2, \cdots : x_i^r = h_i \text{ for } i \geq 1 \rangle.$$

Clearly G is a tree product with vertex groups $A_0 = A$ and $A_i = \langle x_i \rangle$, and edge subgroups $H_{i0} = \langle x_i^r \rangle$ and $H_{0i} = \langle h_i \rangle$. Let $H = \langle h_i : i \geq 1 \rangle$, $K = \langle x_i^r : i \geq 1 \rangle$ and $B = \langle x_i : i \geq 1 \rangle$. By a result of H. Neumann [**3**, Theorem 6.02] the tree product G can also be presented as the amalgamated free product of A and B, amalgamating the proper subgroups H and K. We claim that G is residually finite. Granted that this is so, we can set $h_1 = a_1^2$ and $h_i = a_{i+1}^2 a_i^{-1}$ for $i \geq 2$ to find that H is not profinitely closed in A, since $A/H \cong C_{2^\infty}$ (the Prüfer 2-group).

It remains to show that G is residually finite. To this end it is enough to verify the following:

(a) Each H_i- is profinitely closed in A_i, for $i \geq 0$;
(b) Given any $M \lhd_f A_j$ $(j \geq 0)$ there exists a sequence $(Q_i)_{i \geq 0}$ such that $Q_i \lhd_f A_i$, $|A_i : Q_i|$ is bounded for all i, $Q_0 \cap \langle h_i \rangle = Q_i \cap \langle x_i^r \rangle$ for all $i \geq 1$, and $Q_j = M$.

The residual finiteness of G will then follow from [**5**]. It is well known that condition (a) holds, so we turn to (b).

Suppose $M \lhd_f A$ is given. Set $\langle h_i^{m_i} \rangle = M \cap \langle h_i \rangle$ for $i \geq 1$. Clearly $m_i \leq |A : M|$ for all i, and so the sequence $(P_i)_{i \geq 0}$, where $P_0 = M$ and $P_i = \langle x_i^{rm_i} \rangle$, satisfies (b). Now suppose $L \lhd_f A_i$, where $i \geq 1$, is given, and set $L \cap H_{i0} = \langle x_i^{rs} \rangle$, where s depends on L. Since the factor group $A/\langle h_i^s \rangle$ is residually finite and $\langle h_i \rangle / \langle h_i^s \rangle$ is finite, there exists $M \lhd_f A$ with $M \cap \langle h_i \rangle = \langle h_i^s \rangle$. By the earlier part of this paragraph we can find a sequence (P_i) satisfying (b) such that $P_0 = M$. The sequence Q obtained from P by replacing P_i by L also satisfies (b), and $Q_i = L$. Thus condition (b) also holds, as claimed.

REFERENCES

1. G. Baumslag, *On the residual finiteness of generalized free products of nilpotent groups*, Trans. Amer. Math. Soc. **106** (1963), 193–209.
2. R. J. Gregorac, *On residually finite generalized free products*, Proc. Amer. Math. Soc. **24** (1970), 553–555.
3. H. Neumann, *Generalized free products with amalgamated subgroups*. I, Amer. J. Math. **70** (1948), 590–625.
4. M. Shirvani, *On residually finite HNN-extensions*, Arch. Math. **44** (1985), 110–115.
5. ——, *On residually finite graph products*, J. Pure Appl. Algebra **49** (1987), 281–282.

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