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# FOUR TOPOLOGICALLY EQUIVALENT MEASURES IN THE CANTOR SPACE

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ABSTRACT. We show that one of the binomial numbers discovered by K. J. Huang provides an example of topologically equivalent measures in $2^{\mathbf{N}}$ that are not trivially homeomorphic.

Two Borel measures $\mu$ and $\nu$ in a topological space $X$ are said to be homeomorphic, or topologically equivalent, if $\mu = \nu h$ for some homeomorphism $h : X \to X$. In the Cantor space $X = \{0,1\}^{\mathbf{N}}$, for any $0 \leq r \leq 1$ let $\mu(r) = \pi\mu_n$, where $\mu_n(0) = r$ and $\mu_n(1) = 1 - r$ for all $n$. $\mu(1-r)$ is always homeomorphic to $\mu(r)$, by the mapping $R$ that interchanges 0 and 1 in each coordinate space. When $r \in [0,1]$ is rational, transcendental, or an algebraic integer of degree 2, it is known that no other member of $\{\mu(p) : 0 \leq p \leq 1\}$ is homeomorphic to $\mu(r)$. This is because a number $s \in [0,1]$ is binomially related to such a number $r$ if and only if $s = r$, or $s = 1 - r$ [**2**, **1**]. For each $n > 2$ Huang [**1**] exhibited an algebraic integer $r \in (0,1)$ (and also a noninteger) of degree $n$ that is binomially related to at least one number $s \neq r$, $s \neq 1 - r$. Pinch [**4**] showed that in case $n = 2^{k+1}$ there are at least $2k$ such numbers $s$. However, it has remained an open question whether $\mu(s)$ is homeomorphic to $\mu(r)$ in any of these cases. We shall show that the measures corresponding to Huang's algebraic integer of degree 3 are, in fact, homeomorphic. The other cases remain open.

THEOREM. *Let $r$ be the unique real root of the equation $r^3 + r^2 - 1 = 0$ and let $s = r^2$. Then $\mu(s)$ is homeomorphic to $\mu(r)$, as well as to $\mu(1-r)$ and $\mu(1-s)$.*

For any $u \in \{0,1\}^n$, $n \in \mathbf{N}$, the set $\langle u \rangle$ of points of $X$ whose first $n$ coordinates coincide with $u$ is called a thin cylinder.

LEMMA. *Let $U_i = \langle u_i \rangle$ and $V_i = \langle v_i \rangle$ $(i = 1,2,3)$ be two indexed partitions of $X$ into three thin cylinders, and suppose that*

$$\mu(q)(U_i) = \mu(p)(V_i) \qquad (i = 1,2,3). \tag{1}$$

*Then $\mu(q) = \mu(p)h$ for some homeomorphism $h : X \to X$.*

This is a consequence of a general sufficient condition [**3**, Theorem 3.1] for the existence of a homeomorphism between shift-invariant measures in different spaces of the form $\{1,2,\ldots,k\}^{\mathbf{N}}$. It may be proved directly as follows.

$X$ can be partitioned into three thin cylinders in only two ways, so $U_i$ and $V_i$ must be indexings of either $\{\langle 0 \rangle, \langle 1,0 \rangle, \langle 1,1 \rangle\}$ or $\{\langle 1 \rangle, \langle 0,0 \rangle, \langle 0,1 \rangle\}$. Each $x \in X$

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can be written uniquely as a sequence of blocks $b_1, b_2, \ldots$ equal to $u_1, u_2$, or $u_3$. Let $h(x)$ be the element of $X$ defined by the sequence of blocks $b'_1, b'_2, \ldots$, where $b'_i = v_1, v_2$, or $v_3$, according as $b_i = u_1, u_2$, or $u_3$. Evidently $h$ is bijective. The class $\mathscr{B}$ of thin cylinders of the form $\langle b_1, b_2, \ldots, b_n\rangle$ constitute a base, and so does $h(\mathscr{B})$, because every thin cylinder is the union of at most two members of either class. Hence $h$ is a homeomorphism. Equations (1) and the definition of product measure imply that $\mu(q)(B) = \mu(p)(h(B))$ for each $B \in \mathscr{B}$. Since every open set is a countable disjoint union of members of $\mathscr{B}$ it follows that $\mu(q) = \mu(p)h$.

If we take

$$U_1 = \langle 0\rangle, \qquad U_2 = \langle 1, 0\rangle, \qquad U_3 = \langle 1, 1\rangle$$

and

$$V_1 = \langle 0, 0\rangle, \qquad V_2 = \langle 1\rangle, \qquad V_3 = \langle 0, 1\rangle,$$

then equations (1) become

$$q = p^2, \qquad q(1-q) = 1-p, \qquad (1-q)^2 = p(1-p),$$

which are satisfied by $p = r$, $q = s$. This completes the proof of the theorem. The corresponding homeomorphism $h$ leaves $\langle 1, 0, 0\rangle$ invariant and has four fixed points.

Alternatively, we could take $V_2$, $V_3$, $V_1$ in place of $U_1$, $U_2$, $U_3$. Then equations (1) become

$$1-q = p^2, \qquad q(1-q) = 1-p, \qquad q^2 = p(1-p),$$

which are satisfied by $p = r$, $q = 1-s$. In this case the corresponding homeomorphism permutes $V_3$, $V_2$, $V_1$ cyclically and has period 3. It is equal to $hR$.

It should be noted, however, that $h$ is never unique; if $\mu(q) = \mu(p)h$, then $h$ can always be replaced by $hg$ or $gh$, where $g$ is an arbitrary permutation of the coordinate spaces.

It is easy to verify that all possible choices of $U_i$ and $V_i$ lead to equations (1) that have no solutions other than ones for which $q = p$, $q = 1-p$, or for which $\{p, q\} \subset \{1-r, 1-s, s, r\}$, so no further equivalences can be obtained from the lemma as stated.

## References

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