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FINITE COVERINGS BY NORMAL SUBGROUPS

M. A. BRODIE, R. F. CHAMBERLAIN AND L.-C. KAPPE

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ABSTRACT. B. H. Neumann's characterization of groups possessing a finite covering by proper subgroups and Baer's characterization of groups with finite coverings by abelian subgroups are refined to results about finite coverings by normal subgroups. Various corollaries about the structure of groups having such finite coverings are derived. Using the method employed for the main theorem, a simplified proof of an earlier result of the third author concerning finite coverings by word subgroups is given.

1. Introduction and results. A group is said to be covered by a collection of subgroups if each element of the group belongs to at least one subgroup in the collection. The collection is called a covering of the group [**9**, p. 105]. A covering is called nontrivial if the collection consists of proper subgroups, and trivial otherwise.

In [**8**], B. H. Neumann gives the following characterization of groups having nontrivial finite coverings:

THEOREM A. *A group has a nontrivial finite covering by subgroups if and only if it has a finite noncyclic quotient.*

A characterization of central-by-finite groups in terms of coverings was given by R. Baer as follows:

THEOREM B [**8**, **9**]. *A group is central-by-finite if and only if it has a finite covering consisting of abelian subgroups.*

The main topic of this paper is the characterization of groups which have nontrivial finite coverings by normal subgroups and finite coverings by normal abelian subgroups. Our results are as follows:

THEOREM 1. *A group has a nontrivial finite covering by normal subgroups if and only if it has a quotient isomorphic to an elementary abelian p-group of rank two for some prime p.*

In [**5**], Kontorovič discussed the structure of finite groups with nontrivial normal coverings.

We want to list several corollaries to Theorem 1. Using results from [**1** or **2**], we obtain:

COROLLARY 1. *A nilpotent group has a nontrivial finite covering by subgroups if and only if it has a nontrivial finite covering by normal subgroups.*

Following B. H. Neumann [**8**], we call a covering of a group irredundant if every proper subsystem of the covering fails to cover the group. With this definition we have:

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COROLLARY 2. [1] *All perfect normal subgroups of a group G are contained in every member of an irredundant covering of G by proper normal subgroups.*

The last corollary, an immediate consequence of the preceding one, shows that the problem reduces to finite solvable groups.

COROLLARY 3. *Let $G = \bigcup_{i=1}^{n} N_i$, where $N_1, \dots, N_n$ form an irredundant covering of G by proper normal subgroups. Then G/D with $D = \bigcap_{i=1}^{n} N_i$ is finite and solvable.*

The next result extends Theorem B to finite coverings by abelian normal subgroups.

THEOREM 2. *A group has a finite covering consisting of abelian normal subgroups if and only if it is a central-by-finite two-Engel group.*

Previously, the third author of this paper considered coverings by verbal subgroups; see [**4**]. In the case of finite coverings of this kind, the following result was obtained:

THEOREM 3 [**4**, THEOREM 2]. *Every finite covering of a group consisting of verbal subgroups is trivial.*

The proof of this theorem presented here simplifies the original one by reducing it to the case of finite groups. Also, it removes a gap which seems to appear at one point in the first proof.

We want to mention that this result cannot be extended to finite coverings by characteristic subgroups, since Heineken has constructed finite p-groups of class 2 in which every normal subgroup is characteristic; see [**3**].

As in the case of Theorems A and B, the following result, due to B. H. Neumann, is essential for the proofs of our results.

THEOREM C [**7**, 4.4]. *Let $G = \bigcup_{i=1}^{n} H_i g_i$ where $H_1, \dots, H_n$ are (not necessarily distinct) subgroups of G. Then if we omit from the union any coset $H_i g_i$ for which $[G : H_i]$ is infinite, the union of the remaining cosets is still all of G.*

2. Three preparatory lemmas.

LEMMA 1. *Let $G = H \times M$ where H is a nonabelian simple group. If $N \lhd G$ then $N = H \times S$ or $N = 1 \times S$ where $S \lhd M$.*

PROOF. Since $H, N \lhd G$, we have $[H, N] \subseteq H \cap N \subseteq N$. Now H simple and $[H, N] \lhd H$ imply $[H, N] = H$ or $[H, N] = 1$. Consider $S = \{m \in M;\ \exists\ h \in H, (h, m) \in N\}$. Obviously S is a subgroup of M and $N \subseteq H \times S$.

Let $(h, m) \in H \times S$. Then there exist $k, k' \in H$ such that $(k, m) \in N$ and $k \cdot k' = h$. If $[H, N] = H$, then $H \subseteq N$, so $(k', 1) \in N$ and $(h, m) \in N$. Therefore $N = H \times S$ in this case. Thus $S \lhd M$ since S is a subgroup of M and normal in G.

Now, if $[H, N] = 1$, then $(k, n) \in N$ commutes with all $(h, 1) \in H$. Thus $k \in Z(H)$. Since H is nonabelian and simple, we get $k = 1$ and $N \subseteq M$. So $N = 1 \times S$ and $S \lhd M$. □

A group G is called a PNS-group if every nontrivial proper normal subgroup has a proper normal supplement. Every simple group is a PNS-group in a trivial way.

[1] We want to thank the referee for suggesting Corollaries 2 and 3.

The property of being a PNS-group is not inherited to homomorphic images as can be seen from the following example: Let $G = \prod C_p$ be the unrestricted direct product over all cyclic groups C_p where p ranges over all primes p. Then G is a PNS-group. Let T be its torsion subgroup. The quotient group G/T is a torsion free divisible group which is not a PNS-group. However we have:

LEMMA 2. *Every direct factor of a PNS-group is a PNS-group.*

PROOF. Let $G = L \times M$ be a PNS-group, and let K be a nontrivial proper normal subgroup of M. Then $K \lhd G$ and $K \neq G$. Thus there exists $H \lhd G$, $H \neq G$, such that $KH = G$. Then $K(H \cap M) = G \cap M = M$ and $H \cap M \neq M$ since otherwise $K \subseteq H$, a contradiction. Hence M is a PNS-group. □

For finite groups we have:

LEMMA 3. *A finite group G is a PNS-group if and only if G is the direct product of simple groups.*

PROOF. First let G be a finite PNS-group such that every PNS-group of smaller order is the direct product of simple groups. Without loss of generality we can assume that G is not simple. Let H be a minimal normal subgroup of G. Since G is a PNS-group there exists a proper normal subgroup M of G such that $G = HM$. The minimality of H implies $H \cap M = 1$. Thus $G = H \times M$, and we conclude that H is simple.

Now M is a direct factor of G and of smaller order. Thus, by Lemma 2, M is a PNS-group. It follows by our hypothesis that $M = M_1 \times \cdots \times M_k$ with M_i simple. We conclude that G itself is the direct product of simple groups.

Conversely, assume that G is the direct product of simple groups. If G is abelian, it can be seen easily that G is a PNS-group, since G is the direct product of elementary abelian p-groups. Without loss of generality we can assume that G is not abelian. We prove our claim by induction on the number of simple direct factors of G. Let $G = H \times M$ where H is simple of composite order. Consider $N \lhd G$ with $1 \neq N \neq G$. We can assume $N \neq M, H$. Then, by Lemma 1, there exists $S \lhd M$ with $N = H \times S$ and $S \neq M$, or $N = 1 \times S$ and $1 \neq S$. Now M is the direct product of a smaller number of simple groups than G, hence a PNS-group by our induction hypothesis. Thus there exists $T \lhd M$, $1 \neq T \neq M$, with $M = ST$, and $G = N(1 \times T)$ or $G = N(H \times T)$ respectively. Hence G itself is a PNS-group. □

3. Proof of the theorems and corollaries.

PROOF OF THEOREM 1. First assume $G/N \cong C_p \times C_p$ for some $N \lhd G$. Since $C_p \times C_p$ has a finite nontrivial covering by normal subgroups, it follows by correspondence that G has such a cover.

Conversely, assume that $G = \bigcup_{i=1}^k N_i$, with each N_i a proper normal subgroup of G. Let $N = \bigcap_{i=1}^k N_i$. We may assume that $N_1, \ldots, N_k$ cover G irredundantly and thus by Theorem C $[G : N_i] < \infty$. Hence G/N is finite and $G/N = \bigcup_{i=1}^k N_i/N$, with each N_i/N a proper normal subgroup of G/N. If G/N has a quotient isomorphic to $C_p \times C_p$, so does G. Hence it suffices to show the implication if G is finite.

Suppose that G is a finite group of minimal order with $G = \bigcup_{i=1}^n W_i$, with each W_i a proper normal subgroup of G, but G has no quotient isomorphic to $C_p \times C_p$ for any prime p. To show G is a PNS-group, let $K \lhd G$ and $1 \neq K \neq G$. Then

$G/K = \bigcup_{i=1}^{n} W_iK/K$. Since G was of minimal order it follows that this covering of G/K is trivial. Without loss of generality we can assume that $G = W_1K$. Hence W_1 is a proper normal supplement of K. Thus G is the direct product of simple groups by Lemma 3. If G is abelian we conclude that G is a cyclic group of square-free order. In this case G has no nontrivial finite covering. Thus we can assume that $G = H \times M$ where H is simple of composite order.

Since $|M| < |G|$, there exists $m \in M$ such that $m \notin \mathscr{U} = \bigcup S_i$, where the S_i range over all proper normal subgroups of M. Consider $(h, m) \in G$ with $1 \neq h \in H$. Since G has a nontrivial covering by normal subgroups, there exists a proper normal subgroup N of G and $(h, m) \in N$. We can apply Lemma 1 and obtain $N = 1 \times S$ or $N = H \times S$ with $S \triangleleft M$. Since $h \neq 1$, it follows that $N = H \times S$. But $m \notin \mathscr{U}$ and $m \in S$ imply $S = M$. Hence $N = G$, a contradiction. □

PROOF OF COROLLARY 1. By Theorem 2.6 of [**1**], if G is nilpotent, then G has a finite noncyclic image if and only if G/G' has a finite noncyclic image. By Theorem A, it follows that G admits a finite, nontrivial covering if and only if G/G' does. But every covering of G/G' is a normal covering, and, by correspondence, gives rise to a normal covering of G. □

PROOF OF COROLLARY 2. Let M be a perfect normal subgroup of G. For $M_x = \langle x, M\rangle$, $x \in G$, we observe that M_x/M is cyclic. Hence $M' \subseteq M_x' \subseteq M$. Thus $M_x' = M$, since $M = M'$. By Theorem 1, it follows that M_x has only trivial finite coverings by normal subgroups.

Let $G = \bigcup_{i=1}^{n} N_i$, where $N_{i_1}, \dots, N_n$ are proper normal subgroups of G, covering G irredundantly. Any (normal) covering of G induces a (normal) covering of a subgroup of G by forming intersections of the subgroup with the members of the covering. By the preceding argument, it follows that M_x has only trivial coverings by normal subgroups. Thus M_x is equal to at least one member in the induced covering $M_x = \bigcup_{i=1}^{n}(M_x \cap N_i)$, or $M_x = M_x \cap N_k$ for at least one k, and hence $M \subseteq N_k$. Suppose there exists an N_j in the covering of G such that N_j does not contain M. Then for any $x \in G$, $M \subseteq M_x \subseteq N_k$ where $k \neq j$. Hence $G = \bigcup_{x\in G} M_x \subseteq \bigcup_{i\neq j} N_i$. This is a contradiction, since $N_1, \dots, N_n$ formed an irredundant covering of G. □

PROOF OF COROLLARY 3. Theorem C implies that $H = G/D$ is finite. Therefore the derived series of H becomes stationary after a finite number of steps, i.e. $H^{(k)} = H^{(k+1)}$ for some integer $k \geq 1$. Hence $H^{(k)}$ is a perfect normal subgroup of H. We observe that $H = \bigcup_{i=1}^{n} N_i/D$, where $N_1/D, \dots, N_n/D$ form an irredundant covering of H by proper normal subgroups. By Corollary 2, it follows that $H^{(k)} \subseteq \bigcap_{i=1}^{n} N_i/D$. But $\bigcap_{i=1}^{n} N_i/D = 1$, hence $H^{(k)} = 1$ and $H = G/D$ is solvable. □

PROOF OF THEOREM 2. Assume first that $G = \bigcup_{i=1}^{n} N_i$, with $N_i \triangleleft G$ and N_i abelian. Theorem B implies that $G/Z(G)$ is finite. If $h \in G$, then $h \in N_i$ for some i. Further, $N_i \triangleleft G$ implies $[g, h] \in N_i$ for all $g \in G$. Since N_i is abelian, it follows that $[g, h, h] = 1$. So G is a 2-Engel group.

Now we assume that $G/Z(G)$ is finite and G is 2-Engel. Then $\langle g^G\rangle$ is abelian by [**6**], and $\langle g^G\rangle Z(G)$ is an abelian normal subgroup of G. Choose a transversal $T = \{g_1, \dots, g_n\}$ of $Z(G)$ in G. Then $G = \bigcup_{i=1}^{n} \langle g_i^G\rangle Z(G)$, since each $g \in G$ can be written as g_iz for some $z \in Z(G)$ and some $g_i \in T$. So G has a covering by finitely many abelian normal subgroups. □

The verbal subgroup $\mathscr{W}(G)$ of a group G corresponding to a set of words $\mathscr{W}$ is the subgroup generated by all values in G of the words in $\mathscr{W}$. Without proof we state the following facts about verbal subgroups which will be used in the proof of Theorem 3 without further reference:

(i) $\mathscr{W}(G^\tau) = \mathscr{W}(G)^\tau$ for every homomorphism τ of G;

(ii) $\mathscr{W}(A \times B) = \mathscr{W}(A) \times \mathscr{W}(B)$.

PROOF OF THEOREM 3. Assume the theorem is false. First we show that there exists a finite group for which our claim is not true. Let G be an infinite group having a nontrivial finite covering by verbal subgroups, i.e. $G = \bigcup_{i=1}^m \mathscr{W}_i(G)$ with $\mathscr{W}_i(G)$ properly contained in G for each $1 \leq i \leq m$. By Theorem C, we can assume $[G : \mathscr{W}_i(G)] < \infty$ for all i. Let $N = \bigcap_{i=1}^m \mathscr{W}_i(G)$. Then G/N is finite and $G/N = \bigcup_{i=1}^m \mathscr{W}_i(G/N)$, i.e. G/N has a finite covering by verbal subgroups. Suppose that this covering is trivial. We may assume without loss of generality that $\mathscr{W}_1(G/N) = G/N$. Since $N \subseteq \mathscr{W}_1(G)$, we have $\mathscr{W}_1(G)/N = G/N$, so $G = \mathscr{W}_1(G)$, contradicting our assumption that we had a nontrivial covering of G. Thus, if our theorem fails to be true, there exists a finite group G/N which has a nontrivial covering by verbal subgroups.

Let H be such a group of minimal order having the property that

$$H = \bigcup_{i=1}^n \mathscr{W}_i(H),$$

where $\mathscr{W}_1(H), \ldots, \mathscr{W}_n(H)$ are proper verbal subgroups of H. To show that H is a PNS-group, we consider $K \triangleleft H$ with $1 \neq K \neq H$. Then the resulting covering of H/K by the verbal subgroups $\mathscr{W}_i(H/K)$, $i = 1, \ldots, n$, is trivial since $|H/K| < |H|$. It follows that $H = K \cdot \mathscr{W}_i(H)$ for some i, hence $\mathscr{W}_i(H)$ is the proper normal supplement of K in H. Thus, by Lemma 3, $H = H_1 \times \cdots \times H_k$ with H_i simple, for $i = 1, \ldots, k$. For any verbal subgroup $\mathscr{W}(H)$ of H we have $\mathscr{W}(H) = \mathscr{W}(H_1) \times \cdots \times \mathscr{W}(H_k)$. The simplicity of each H_i implies that $\mathscr{W}(H)$ is a proper verbal subgroup of H if and only if $\mathscr{W}(H_i) = 1$ for at least one H_i. This implies that any $h \in H$ with nonidentity components for each i is contained in no proper verbal subgroup of H, contradicting our assumption that H had a nontrivial covering by verbal subgroups. Hence no finite group, and therefore no group, has a nontrivial covering by verbal subgroups. $\square$

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DEPARTMENT OF MATHEMATICAL SCIENCES, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON, BINGHAMTON, NEW YORK 13901 (Current address of L. C. Kappe and M. A. Brodie)

Current address: (R. F. Chamberlain): Supercomputing Research Center, 4380 Forbes Boulevard, Lanham, Maryland 20706