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UNIQUE SOLUTIONS FOR A CLASS OF DISCONTINUOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with the Cauchy Problem

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \in \mathbf{R}^n,$$

where the vector field f may be discontinuous with respect to both variables t, x. If the total variation of f along certain directions is locally finite, we prove the existence of a unique solution, depending continuously on the initial data.

1. Introduction. Let f be a vector field on $\mathbf{R}^n$. By definition, a Carathéodory solution of the Cauchy Problem

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0 \in \mathbf{R}^n, \tag{1.1}$$

is an absolutely continuous function $t \to x(t)$ which takes the value x_0 at $t = t_0$ and satisfies the differential equation in (1.1) at almost every t. If f is not continuous, Peano's theorem does not apply and (1.1) may not have any solution. Some authors have thus introduced new definitions of generalized or relaxed solutions for (1.1), for which a satisfactory existence theorem could then be proven [**5**, **6**, **9**]. An alternative approach to discontinuous O.D.E.'s, pursued in [**4**, **7**, **8**], relies on the study of certain conditions on f which are weaker than continuity, yet sufficient to guarantee the existence of Carathéodory solutions. This led to the investigation of directional continuity. For a fixed $M > 0$, consider the cone

$$\Gamma^M = \{(t, x) \in \mathbf{R}^{n+1}; \|x\| \leq Mt\}. \tag{1.2}$$

We say that a map $f\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ is Γ^M-continuous if, for every $(t_0, x_0) \in \mathbf{R}^{n+1}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$t_0 \leq t < t_0 + \delta, \qquad \|x - x_0\| \leq M(t - t_0) \Rightarrow \|f(t, x) - f(t_0, x_0)\| < \varepsilon.$$

Assuming that $\|f(t, x)\| \leq L < M$ for all t, x, solutions of O.D.E.'s with Γ^M-continuous right-hand sides were obtained in [**7**] as limits of polygonal approximations, in [**2**] through an application of Schauder's fixed point theorem, and in [**3**] by means of an upper semicontinuous, convex-valued regularization. These results acquire additional interest in connection with the theory of multivalued differential equations. Indeed, the existence of directionally continuous selections for lower semicontinuous multifunctions now provides a very effective tool for the study of differential inclusions [**2**, **3**].

The present paper is concerned with the problem of uniqueness and continuous dependence. In the classical theory, the uniqueness of solutions of (1.1) is proved assuming that f is locally Lipschitz continuous. Here we consider a much weaker

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condition, which does not imply the continuity of f. Let $\prec$ be the partial ordering on $\mathbf{R}^{n+1}$ induced by the cone Γ^M:

$$(t,x) \prec (t',x') \quad \text{iff } \|x'-x\| \le M(t'-t). \tag{1.3}$$

Using this ordering, one can define a class of vector fields with bounded "directional variation".

DEFINITION 1. A vector field $f\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ has bounded Γ^M-variation if there exists a constant C such that

$$\sum_{i=1}^{N} \|f(t_i,x_i) - f(t_{i-1},x_{i-1})\| \le C$$

for every finite sequence (t_i,x_i), $i = 0,1,\dots,N$, with $(t_0,x_0) \prec (t_1,x_1) \prec \cdots \prec (t_N,x_N)$.

DEFINITION 2. A vector field $f\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ has locally bounded Γ^M-variation if, for every $(t_0,x_0) \in \mathbf{R}^{n+1}$, there exist $\delta > 0$ and a constant C such that

$$\sum_{i=1}^{N} \|f(t_i,x_i) - f(t_{i-1},x_{i-1})\| \le C \tag{1.4}$$

for every finite sequence (t_i,x_i), $i = 1,\dots,N$, satisfying

$$(t_0,x_0) \prec (t_1,x_1) \prec \cdots \prec (t_N,x_N), \qquad t_N < t_0 + \delta. \tag{1.5}$$

Our main result shows that if f has locally bounded directional variation, then the solution of (1.1) is unique:

THEOREM 1. *Let $f\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ be a vector field with locally bounded Γ^M-variation. If $\|f(t,x)\| \le L < M$ for all t,x, then the Cauchy Problem* (1.1) *has a unique forward solution $x(\cdot)$, which is defined on $[t_0,\infty)$. Moreover, the restriction of $x(\cdot)$ to any bounded interval $[t_0,T]$ depends continuously on the initial value x_0.*

In §2 we establish an intermediate result. The proof of Theorem 1 is then completed in §3; it relies on the construction of a directionally continuous version of f and on a proper use of the classical Contraction Mapping Principle.

2. An auxiliary theorem. The following uniqueness result for solutions of the Cauchy Problem

$$\dot{x}(t) = f(t,x(t)), \qquad x(0) = 0 \in \mathbf{R}^n, \tag{2.1}$$

will be instrumental for the proof of Theorem 1.

THEOREM 2. *Assume that there exist constants $L, M, \delta > 0$ and a function $\phi\colon \Gamma^M \to \mathbf{R}$ such that*

(i) $\|f(t,x)\| \le L < M$,

(ii) $\lim_{t\to 0+,\|x\|\le Mt} \phi(t,x) = \phi(0,0) = 0$,

(iii) $\|f(t,x) - f(s,y)\| \le \phi(t,x) - \phi(s,y)$ *whenever* $(t,x),(s,y) \in \Gamma^M$, $0 \le s \le t < \delta$, $\|x-y\| \le M(t-s)$.

Then (2.1) *has a unique forward solution, defined on some positive interval* $[0,T]$.

PROOF. Using (ii), choose $T \in (0,\delta/2)$ such that

$$\phi(t,x) \le (M-L)/4 \tag{2.2}$$

whenever $0 \le t \le 2T$, $\|x\| \le Mt$. Consider the set $\mathbf{K}$ of all continuous mappings $y\colon [0,T] \to \mathbf{R}^n$ with $y(0) = 0$ and with Lipschitz constant L. Define the Picard operator $P\colon K \to K$ by setting

$$P(y)(t) = \int_0^t f(s, y(s))\, ds. \tag{2.3}$$

For any $y \in \mathbf{K}$, (iii) and (2.2) imply that the map $s \to f(s, y(s))$ has bounded variation, hence the integral in (2.3) is well defined. Clearly $P(y) \in K$ because of (i). We claim that P is a strict contraction. For any $y_1, y_2 \in K$, set

$$\sigma = \|y_1 - y_2\|_{\mathscr{C}^0} = \max_{t\in[0,T]} \|y_1(t) - y_2(t)\| \le 2LT. \tag{2.4}$$

Call $\xi = \sigma/2M$ and define the auxiliary map

$$z(t) = [y_1(t-\xi) + y_2(t-\xi)]/2, \qquad t \in [\xi, T+\xi]. \tag{2.5}$$

Observe that $\xi \le T$ and that z has Lipschitz constant L. Moreover

$$\|z(t+\xi) - y_i(t)\| \le \sigma/2 = M[(t+\xi) - t], \tag{2.6}$$

therefore we can apply (iii) and deduce

$$\|f(t+\xi, z(t+\xi)) - f(t, y_i(t))\| \le \phi(t+\xi, z(t+\xi)) - \phi(t, y_i(t)) \tag{2.7}$$

for all $t \in [0,T]$, $i = 1, 2$. Observe that (iii) trivially implies

$$\|x - y\| \le M(t-s) \Rightarrow \phi(s,y) \le \phi(t,x). \tag{2.8}$$

In particular, the maps $t \to \phi(t, y_i(t))$ and $t \to \phi(t+\xi, z(t+\xi))$ are nondecreasing. For $0 \le r \le (M-L)/4$, $i = 1, 2$, define

$$\begin{aligned} \tau_i(r) &= \inf\{t \in [0,T]; \phi(t, y_i(t)) \ge r\} \wedge T, \\ \tau_z(r) &= \inf\{t \in [0,T]; \phi(t+\xi, z(t+\xi)) \ge r\} \wedge T. \end{aligned}$$

We claim that

$$0 \le \tau_i(r) - \tau_z(r) \le \sigma/(M-L). \tag{2.9}$$

Indeed, the first inequality follows from (2.6), (2.8) which imply

$$\phi(t, y_i(t)) \le \phi(t+\xi, z(t+\xi)) \quad \forall t \in [0,T].$$

To prove the second inequality, set $\tau = \tau_z(r)$. If $\tau = T$, the conclusion is obvious. Otherwise, for any $\tau' \in (\tau, T]$, the choice of ξ implies

$$\begin{aligned} &\left\| z(\tau'+\xi) - y_i\left(\tau' + \frac{\sigma}{M-L}\right)\right\| \\ &\le \|z(\tau'+\xi) - y_i(\tau')\| + \left\| y_i(\tau') - y_i\left(\tau' + \frac{\sigma}{M-L}\right)\right\| \\ &\le \frac{\sigma}{2} + \frac{\sigma}{M-L}\cdot L = M\left(\frac{\sigma}{M-L} - \xi\right). \end{aligned}$$

Therefore (2.8) yields

$$r \le \phi(\tau'+\xi, z(\tau'+\xi)) \le \phi\left(\tau' + \frac{\sigma}{M-L}, y_i\left(\tau' + \frac{\sigma}{M-L}\right)\right),$$

hence $\tau_i(r) \leq \tau' + \sigma/(M-L)$ for all $\tau' > \tau$, completing the proof of (2.9). Relying on (2.2) and (2.9) we now obtain our basic estimate, through a change in the order of integration:

$$\begin{aligned}
&\int_0^T \|f(t+\xi, z(t+\xi)) - f(t, y_i(t))\|\, dt \\
&\quad \leq \int_0^T [\phi(t+\xi, z(t+\xi)) - \phi(t, y_i(t))]\, dt = \int_0^T \int_{\phi(t,y_i(t))}^{\phi(t+\xi,z(t+\xi))} dr\, dt \\
&\quad = \int_0^{(M-L)/4} [\tau_i(r) - \tau_z(r)]\, dr \leq \frac{M-L}{4} \cdot \frac{\sigma}{M-L} = \frac{\sigma}{4}.
\end{aligned}$$

From (2.10) it follows that

$$\int_0^T \|f(t, y_1(t)) - f(t, y_2(t))\|\, dt \leq \frac{\sigma}{2} = \frac{\|y_1 - y_2\|}{2}.$$

Therefore the Picard operator P is a strict contraction on K and has a unique fixed point $x(\cdot)$, which yields the unique Carathéodory solution of (2.1) on $[0, T]$.

3. Proof of Theorem 1. For simplicity we assume $t_0 = 0$, $x_0 = 0 \in \mathbf{R}^n$, which is not restrictive. Define the auxiliary vector field $\tilde{f}\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ by setting

$$\tilde{f}(t,x) = \lim_{s \to t^+} f(s,x). \tag{3.1}$$

The limit in (3.1) always exists because, for $\varepsilon > 0$ small enough, the map $s \to f(s,x)$ has bounded variation on $[t, t+\varepsilon)$. Set $\widetilde{M} = (M+L)/2$. For all (t,x) inside the cone

$$\Gamma^{\widetilde{M}} = \{(t,x);\, \|x\| \leq \widetilde{M}t\}$$

define

$$\phi(t,x) = \sup\left\{\sum_{i=1}^{N} \|\tilde{f}(t_i, x_i) - \tilde{f}(t_{i-1}, x_{i-1})\|\right\}, \tag{3.2}$$

the supremum being taken over all finite sequences $\{(t_i, x_i); i = 0, \ldots, N\}$ with $N \geq 1$, $(t_0, x_0) = (0,0)$, $(t_N, x_N) = (t,x)$ and $\|x_i - x_{i-1}\| \leq \widetilde{M}(t_i - t_{i-1})$ for all i. Some properties of $\tilde{f}$ and ϕ will be examined in the next lemmas.

LEMMA 1. *The vector field $\tilde{f}$ is $\Gamma^{\widetilde{M}}$-continuous.*

PROOF. Assume, on the contrary, that there exists a sequence of points (t_n, x_n) converging to (t,x), with $\|x_n - x\| \leq \widetilde{M}(t_n - t)$ but

$$\|\tilde{f}(t_n, x_n) - \tilde{f}(t,x)\| > \eta > 0 \quad \forall n \geq 1. \tag{3.3}$$

By (3.1), we can choose $t_n' \in (t_n, t_n + 1/n]$ suitably close to t_n and still have

$$\|f(t_n', x_n) - \tilde{f}(t,x)\| > \eta \quad \forall n \geq 1. \tag{3.4}$$

Let $\{s_m\}$ be a sequence strictly decreasing to t, with the property

$$\|f(s_m, x) - \tilde{f}(t,x)\| < \eta/2 \quad \forall m \geq 1. \tag{3.5}$$

Let $\delta > 0$ be given. By induction, we now construct a sequence (τ_j, y_j) such that, recalling (1.3),

$$0 < \tau_j < \delta, \quad (\tau_{j+1}, y_{j+1}) \prec (\tau_j, y_j) \ \forall j \geq 1, \tag{3.6}$$

$$\begin{cases} \text{if } j \text{ is even,} & (\tau_j, y_j) = (t'_{n_j}, x_{n_j}) \quad \text{for some } n_j, \\ \text{if } j \text{ is odd,} & (\tau_j, y_j) = (s_{m_j}, x) \quad \text{for some } m_j. \end{cases} \tag{3.7}$$

This can be done as follows. First choose m_1 such that $s_{m_1} < \delta$ and set $(\tau_1, y_1) = (s_{m_1}, x)$. If (τ_j, y_j) has been defined for all $j < 2k$, select n_{2k} so large that $\widetilde{M} t'_{n_{2k}} \leq M(\tau_{2k-1} - t'_{n_{2k}})$ and set $(\tau_{2k}, y_{2k}) = (t'_{n_{2k}}, x_{n_{2k}})$. Then select m_{2k+1} so large that $\|y_{2k}\| = \|x_{n_{2k}}\| \leq M(\tau_{2k} - s_{m_{2k+1}})$ and set $(\tau_{2k+1}, y_{2k+1}) = (s_{m_{2k+1}}, x)$. All this is possible because the sequences $\{t'_n\}, \{s_m\}$ converge to t and $\|x_n - x\| < M(t'_n - t)$ for every n. The sequence $\{(\tau_j, y_j); j \geq 1\}$ then satisfies (3.6), (3.7). From (3.4) and (3.5) it follows that

$$\sum_{j=1}^{\infty} \|f(\tau_{j+1}, y_{j+1}) - f(\tau_j, y_j)\| = \infty, \tag{3.8}$$

hence the vector field f cannot have locally bounded Γ^M-variation near the point (t, x). This contradiction proves the lemma.

LEMMA 2. *The sets of Carathéodory solutions for the differential equations $\dot{x}(t) = f(t, x(t))$ and $\dot{x}(t) = \tilde{f}(t, x(t))$ coincide.*

PROOF. Let $u\colon [a, b] \to \mathbf{R}^n$ be any continuous map with Lipschitz constant L. Define J as the set of times $t \in [a, b]$ for which there exists some sequence $\{t_k\}$, strictly decreasing to t, with $f(t_k, u(t_k))$ converging to $f(t, u(t))$. The measurability of the map $t \to f(t, u(t))$ implies that $\text{meas}(J) = b - a$ (see Lemma 2.3 in [**3**] for details). Since $L < \widetilde{M}$, the $\Gamma^{\widetilde{M}}$-continuity of $\tilde{f}$ implies $\tilde{f}(t, u(t)) = f(t, u(t))$ for all $t \in J$, hence almost everywhere on $[a, b]$. Lemma 2 is now clear, because every integral curve for the vector fields f or $\tilde{f}$ is Lipschitz continuous with constant L.

LEMMA 3. *There exists $\delta > 0$ such that ϕ is bounded on the set $\Delta = \{(t, x); \|x\| \leq \widetilde{M} t, 0 \leq t < \delta\}$. Moreover,*

$$\|\tilde{f}(t, x) - \tilde{f}(s, y)\| \leq \phi(t, x) - \phi(s, y) \tag{3.9}$$

for all $(t, x), (s, y) \in \Delta$ with $\|x - y\| \leq \widetilde{M}(t - s)$;

$$\lim_{\substack{t \to 0^+ \\ \|x\| \leq \widetilde{M} t}} \phi(t, x) = 0. \tag{3.10}$$

PROOF. We begin by proving the last assertion. If (3.10) fails, then there exists a constant $\eta > 0$ such that, for every $\varepsilon > 0$, one can find a finite sequence $\{(t_i, x_i); i = 0, \ldots, N\}$ satisfying

$$(t_0, x_0) = (0, 0), \quad t_N < \varepsilon, \quad \|x_i - x_{i-1}\| \leq \widetilde{M}(t_i - t_{i-1}), \tag{3.11}$$

$$\sum_{i=1}^{N} \|\tilde{f}(t_i, x_i) - \tilde{f}(t_{i-1}, x_{i-1})\| > \eta. \tag{3.12}$$

Choosing $\tau_i > t_i$ suitably close to t_i, we obtain a finite sequence (τ_i, x_i) which satisfies

$$(3.13)\quad 0 < \tau_0 < \cdots < \tau_N < \varepsilon, \quad \|x_i\| \leq \widetilde{M}\tau_i, \quad \|x_i - x_{i-1}\| \leq M(\tau_i - \tau_{i-1}),$$

$$(3.14)\qquad \sum_{i=1}^{N} \|f(\tau_i, x_i) - f(\tau_{i-1}, x_{i-1})\| > \eta.$$

Let $\delta > 0$ be given. Construct a sequence of finite sets $S_k = \{(\tau_i^k, x_i^k); i = 0, \dots, N_k\}$, $k \geq 1$, inductively as follows. Define S_1 to be any finite sequence for which (3.13) and (3.14) hold with $\varepsilon = \delta$. When S_{k-1} has been defined, let S_k be any finite sequence which satisfies (3.13) and (3.14) with $\varepsilon = \tau_0^{k-1}(M + \widetilde{M})/(M - \widetilde{M})$. Since $\|x_0^{k-1}\| \leq \widetilde{M}\tau_0^{k-1}$, $\|x_{N_k}^k\| \leq \widetilde{M}\tau_{N_k}^k$, this choice of ε implies

$$(3.15)\qquad (\tau_{N_k}^k, x_{N_k}^k) \prec (\tau_0^{k-1}, x_0^{k-1}) \quad \forall k > 1.$$

Because of (3.15), the set $S = \bigcup_{k\geq 1} S_k$ is totally ordered by the relation $\prec$ defined at (1.3). We can thus arrange its elements into a unique decreasing sequence, say $S = \{(\tau_j, x_j), j \geq 0\}$, with $(\tau_j, x_j) \prec (\tau_{j-1}, x_{j-1})$ for all $j \geq 1$. Since every S_k satisfies (3.14), it follows that

$$(3.16)\qquad \sum_{j=1}^{\infty} \|f(\tau_j, x_j) - f(\tau_{j-1}, x_{j-1})\| = \infty.$$

By (3.16), f cannot have locally bounded Γ^M-variation near the point $(0,0)$. This contradiction proves (3.10). The existence of a set Δ where ϕ is bounded is an obvious consequence of (3.10). The "dynamic programming" equation (3.9) follows easily from the definition of ϕ at (3.2).

It is now possible to complete the proof of Theorem 1. By Lemma 3, one can apply Theorem 2 to the vector field $\tilde{f}$ (with M replaced by $\widetilde{M}$) and obtain the existence of a unique local solution $x(\cdot)$ for the Cauchy Problem

$$(3.17)\qquad \dot{x}(t) = \tilde{f}(t, x(t)), \qquad x(t_0) = x_0,$$

on some forward interval $[t_0, T]$. By Lemma 2, $x(\cdot)$ is also the unique solution of (1.1) on $[t_0, T]$. Since f is bounded, this solution can be uniquely extended to $[t_0, \infty)$. To prove the continuous dependence on the initial data, consider the upper semicontinuous compact convex valued multifunction F:

$$(3.18)\qquad F(t, x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\{f(s, y); |s - t| < \varepsilon, \|y - x\| < \varepsilon\},$$

where $\overline{\mathrm{co}}$ stands for closed convex hull. By Lemma 3.2 in [**3**], the Carathéodory solutions of (1.1) coincide with those of the multivalued Cauchy Problem

$$(3.19)\qquad \dot{x}(t) \in F(t, x(t)), \qquad x(t_0) = x_0.$$

For any fixed $T > t_0$, let $\mathscr{T}(x_0) \subseteq \mathscr{C}([t_0, T]; \mathbf{R}^n)$ denote the family of solutions of (3.19). The set-valued map $x_0 \to \mathscr{T}(x_0)$ is then upper semicontinuous [**1**, p. 104]. By uniqueness, in our case $\mathscr{T}(x_0)$ reduces to a single element, therefore it depends continuously on x_0.

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