PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 3, November 1988

NONEXPANSIVE ACTIONS OF TOPOLOGICAL SEMIGROUPS ON STRICTLY CONVEX BANACH SPACES AND FIXED POINTS

WOJCIECH BARTOSZEK

(Communicated by John B. Conway)

ABSTRACT. Let C be a closed convex subset of a strictly convex Banach space X and $\{T_s: s \in S\}$ be a continuous representation of a semitopological semigroup S as nonexpansive mappings of C into itself. The main result establishes the fact that if for some $x \in C$ the trajectory $\{T_s x: s \in S\}$ is relatively compact and $AP(S)$ has a left invariant mean then $K = \overline{\operatorname{conv}\{T_s x: s \in S\}}$ contains a common fixed point for $\{T_s\}_{s \in S}$.

Let C be a closed, convex subset of a Banach space X, and T be a nonexpansive mapping of C into C (i.e., $||Tx - Ty|| \leq ||x - y||$ for every $x, y \in C$). For $x \in C$ the orbit of x is the set $\mathscr{O}(x) = \{T^n x: n \geq 0\}$ and the ω-limit set of x is defined by $\omega(x) = \{y: \lim T^{n_k} x = y$ for some subsequence $n_k \nearrow \infty\}$. The ω-limit set of a point x is easily shown to be closed and T-invariant although possibly empty. If $\omega(x)$ is nonempty then it is minimal (the orbit $\mathscr{O}(y)$ is a dense subset of $\omega(x)$ for every $y \in \omega(x)$) and the action of T on $\omega(x)$ is isometric (see [**D.S.**]). By [**R.S.**], nonempty $\omega(x)$ can be given the structure of a monothetic group. So there exists a T-invariant probability measure μ (μ is invariant if $\mu \circ T^{-1} = \mu$) on $\omega(x)$ if and only if $\omega(x)$ is (nonempty) compact (see also [**B.D.**]). The following lemma gives connections between the existence of compact orbits and T-invariant measures in slightly more general situations.

LEMMA 1. *Let X, C, T be as above. If C is separable and μ is a T-invariant probability (on a Borel σ-field) then every $x \in \operatorname{supp}\mu$ is recurrent and $\omega(x)$ is compact* ($\operatorname{supp}\mu$ *denotes here the smallest closed subset of C of full measure μ*).

Proof. Notice that $\operatorname{supp}\mu$ is a T-invariant subset of C (i.e., if $x \in \operatorname{supp}\mu$ then $Tx \in \operatorname{supp}\mu$). By the classical Poincaré recurrence theorem, the set of recurrent points is full measure, so the set of recurrent points is dense in $\operatorname{supp}\mu$. Since T is nonexpansive, every $x \in \operatorname{supp}\mu$ is recurrent. Thus $\operatorname{supp}\mu = \bigcup_{x \in \operatorname{supp}\mu} \omega(x)$, and for every $x, y \in \operatorname{supp}\mu$ the limit sets $\omega(x), \omega(y)$ coincide or are disjoint. In order to show compactness of $\omega(x)$ it is enough to show the existence of T-invariant probability on $\omega(x)$. Let $\mathscr{T}$ denote the partition of $W = \operatorname{supp}\mu$ on the sets $\omega(x)$. It is known (see [**R**] or [**P**]) that there exists a system of canonical measures μ_τ concentrated on τ ($\tau = \omega(x)$ for $x \in W$) such that

$$\mu = \int_{W/\mathscr{T}} \mu_\tau \nu(d\tau)$$

Received by the editors January 30, 1987 and, in revised form, November 30, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47H10, 47H20; Secondary 47H09.

and ν is some probability measure on $W/\mathscr{T}$. Let $\sigma_{\mathscr{T}}$ denote the sub σ-field generated by sets of the partition $\mathscr{T}$. Clearly

$$\begin{aligned}\int f\,d\mu_{\omega(x)} &= E(f\,|\,\sigma_{\mathscr{T}})(x) = E(f\,|\,\sigma_{\mathscr{T}})(Tx)\\ &= E(f\circ T\,|\,\sigma_{\mathscr{T}})(x) = \int f\circ T\,d\mu_{\omega(x)},\end{aligned}$$

for ν almost all $\omega(x)\in W/\mathscr{T}$ (f is here an arbitrary continuous function on $\operatorname{supp}\mu$). Thus, for ν and almost all τ, the measures μ_τ are T-invariant, so $\omega(x)$ is compact for x from a dense subset of W (see [**R.S.**] or [**B.D.**]). But the set $\{y: \omega(y)$ is compact$\}$ is closed (and convex if X is strictly convex), and thus for all $x\in\operatorname{supp}\mu$ the orbit $\omega(x)=\overline{\mathscr{O}(x)}$ is compact.

PROPOSITION. *Let X be a strictly convex Banach space and C be a separable, convex, closed subset of X. If $T: C\to C$ is nonexpansive then the following conditions are equivalent:*

(i) *the set $F(T)$ of fixed points of T is nonempty;*

(ii) *there exists a T-invariant probability;*

(iii) *there exists $x\in C$ such that $\overline{\mathscr{O}(x)}$ is compact.*

Proof. Because (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(ii) are trivial and (ii)$\Rightarrow$(iii) follows from our lemma we only have to show (ii)$\Rightarrow$(i). Let μ be the T-invariant probability measure on $\omega(x)$ (unique by nonexpansiveness of T). Since T is affine on $\overline{\operatorname{conv}\omega(x)}$ (see [**E**] or [**Y**]), for every $x^*\in X^*$, $x\circ T$ is affine, continuous and $x^*(\operatorname{bar}\mu) = \int_{\omega(x)} x^*(y)\mu(dy) = \int_{\omega(x)} x^*\circ T(y)\mu(dy) = x^*\circ T(\operatorname{bar}\mu) = x^*(T(\operatorname{bar}\mu))$. Since the functionals separate points of X the barycenter of μ is a fixed point of T.

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a\in S$ the mappings $s\to as$ and $s\to sa$ from S to S are continuous. Let X be a strictly convex Banach space, and $S\ni s\to T_s$ be a continuous representation of S as nonexpansive mappings on a closed convex subset C of X into C, i.e., $T_{ab}(x)=T_a(T_b(x))$, $a,b\in S$, $x\in C$, and the mapping $s\to T_s x$ from S into C is continuous for every $x\in C$. If $f\in C(S)$ and $a\in S$ define $l_a f(s)=f(as)$ and $r_a f(s)$ for all $s\in S$ ($C(S)$ denotes here the set of all continuous bounded functions on S). Recall, a function $f\in C(S)$ is said to be almost periodic on S if $\{r_a f: a\in S\}$ is relatively compact in the norm topology of $C(S)$. The subalgebra of all almost periodic functions on S we denote by $AP(S)$. A linear functional $m\in AP(S)^*$ is called a left invariant mean if for all $a\in S$ and $f\in AP(S)$ we have $\langle l_a f, m\rangle = \langle f, m\rangle$ and $\langle 1, m\rangle = 1$. The following theorem is a partial solution of Problem 2 from [**L.2**].

THEOREM. *Let $\{T_s: s\in S\}$ be a continuous representation of a semitopological semigroup S as nonexpansive mappings on a closed convex subset of a strictly convex Banach space X. If $AP(S)$ has a left invariant mean, $x\in C$ such that $\{T_s x: s\in S\}$ is relatively compact then $K=\overline{\operatorname{conv}}\{T_s x: s\in S\}$ contains a common fixed point for $\{T_s\}_{s\in S}$.*

Proof. It is clear that for every continuous function f on C the function $\tilde{f}$ defined on S as $\tilde{f}(s)=f(T_s x)$ belongs to $AP(S)$. Thus a left invariant mean m on $AP(S)$ defines a probability measure μ on $\overline{\{T_s x: s\in S\}}$. Clearly (see [**L.1**]), the measure μ

is T_s-invariant for every $s \in S$. By Lemma 1, if $y \in \operatorname{supp}\mu$ then for every $s \in S$, y is T_s recurrent and T_s is affine on $K' = \overline{\operatorname{conv}(\operatorname{supp}\mu)} \subseteq K$. But by our proposition the barycenter of measure $\operatorname{bar}(\mu) \in K'$ is a fixed point for every T_s.

It is a pleasure to thank Professors Lau and Sine for sending me preprints of some of their works. These lead to the following remark: Problem 2 of Lau [**L.2**] has a negative answer in general. There is an appropriate nonexpansive map T in a 3-dimensional (Banach) space for which $(N+1)^{-1}(I+T+\cdots+T^N)x$ converges, but the limit is not a fixed point, and there is no fixed point in the closed convex hull of the orbit (see Robert Sine, *Behaviour of iterates in the Poincare metric*, preprint, 1986).

REFERENCES

[**B.D.**] W. Bartoszek and T. Downarowicz, *Compactness of trajectories of dynamical systems in uniform complete spaces*, Suppl. Rend. Circ. Mat. Palermo (The 13th Winter School on Abstract Analysis, Srni, 1985).

[**D.S.**] C. M. Dafermos and M. Slemrod, *Asymptotic behaviour of nonlinear contraction semigroup*, J. Funct. Anal. **13** (1973), 97–106.

[**E**] M. Edelstein, *On non-expansive mappings of Banach spaces*, Proc. Cambridge Philos. Soc. **60** (1964), 439–447.

[**L.1**] A. T. Lau, *Invariant means on almost periodic functions and fixed point properties*, Rocky Mountain J. Math. **3** (1973), 69–76.

[**L.2**] A. T. Lau, *Semigroup of nonexpansive mappings on a Hilbert space*, J. Math. Anal. Appl. **105** (1985), 514–522.

[**P**] W. Parry, *Entropy and generators in ergodic theory*, Yale Univ. Press, New Haven, Connecticut, 1966.

[**R**] W. A. Rochlin, *Isbrannyje woprosy mietriczieskoj teorii dynamichieskich sistiem*, Uspekhi Mat. Nauk **30** (1949), 57–128.

[**R.S.**] S. Roehring and R. C. Sine, *The structure of ω-limit sets of nonexpansive maps*, Proc. Amer. Math. Soc. **81** (1981), 398–400.

[**Y**] C-L. Yen, *On the rest points of a nonlinear nonexpansive semigroup*, Pacific J. Math. **45** (1973), 699–706.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, WROCŁAW BRANCH, KOPERNIKA 18, 51–617 WROCŁAW, POLAND

Current address: ul. Sułkowskiego 4 m. 14, 85-655 Bydgoszcz, Poland