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THE WIENER LEMMA AND COCYCLES

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ABSTRACT. We give a sufficient and necessary condition for a function with its values in the unit circle to be a multiplicative coboundary. This theorem generalizes the following result of Veech [1]. Let $T\colon \mathbf{T} \to \mathbf{T}$ be a rotation of the unit circle $\mathbf{T}$ by an irrational angle θ. Let $F\colon \mathbf{T} \to \mathbf{T}$ be a measurable function. Then F is a multiplicative coboundary iff

$$\int_{\mathbf{T}} F(x)F(Tx)\cdots F(T^{n-1}x)\,d\mu(x) \to 1, \quad \text{as } ||n\theta|| \to 0,$$

where $||n\theta||$ is the distance of $n\theta$ from integers and μ is the Haar measure.

Let $T\colon (X,\Sigma,\mu) \to (X,\Sigma,\mu)$ be a measure preserving automorphism which is ergodic. Let $F\colon X \to \mathbf{T}$ be a measurable function. We call F a coboundary $\operatorname{mod}\lambda$ if there is a function $g\colon X \to \mathbf{T}$ and $\lambda \in \mathbf{T}$ such that $F(x) = \lambda g(x)/g(Tx)$ a.e. For every integer $n \geq 0$, let $F_n(x) = F(x)F(Tx)\cdots F(T^{n-1}x)$. For $n < 0$ we set $F_n(x) = F_{-n}(T^n x)$.

DEFINITION. Let $A \subset \mathbf{Z}$. The upper density of A is the number

$$\overline{\lim_{n\to\infty}} \frac{1}{2n+1}\operatorname{card}([-n,n]\cap A).$$

THEOREM 1. *The following two conditions are equivalent*:

(i) *F is a coboundary* $\operatorname{mod}\lambda$.

(ii) *There is a function $f \in L^2(X,\Sigma,\mu)$ and $\varepsilon > 0$ such that the set*

$$\left\{n \in \mathbf{Z}\colon \left|\int_{\mathbf{T}} F_n(x)f(T^n x)\overline{f(x)}\,d\mu(x)\right| > \varepsilon\right\}$$

has positive upper density.

PROOF. This theorem is an easy corollary of Wiener's Theorem. Let

$$U\colon L^2(X,\Sigma,\mu) \to L^2(X,\Sigma,\mu)$$

be a unitary operator defined as follows:

$$Uf(x) = F(x)f(Tx).$$

Then the condition of being a coboundary $\operatorname{mod}\lambda$ is equivalent to the existence of $g \in L^2(X,\Sigma,\mu)$ such that $Ug = \lambda g$. In fact, if such g exists then $|g\circ T| = |g|$ and by ergodicity the modulus of g is constant a.e., so we can assume that it is equal to 1.

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We invoke Wiener's Theorem now (see [2]):

$$\frac{1}{2n+1}\sum_{k=-n}^{n}|(U^k f, f)|^2 \to \sum_{\lambda\in\mathbf{T}}|\sigma_f(\{\lambda\})|^2$$

where σ_f is the spectral measure of f. This means that the set of the eigenvalues of U is nonempty iff for some $\varepsilon > 0$ and some $f \in L^2(X, \Sigma, \mu)$ the set $\{n \in \mathbf{Z}\colon |(U^n f, f)| > \varepsilon\}$ has positive upper density. This is exactly condition (ii). □

COROLLARY. *Suppose there is an $\varepsilon > 0$ such that the set:*

$$\left\{n \in \mathbf{Z}\colon \left|\int_{\mathbf{T}} F_n(x)\,d\mu(x)\right| > \varepsilon\right\}$$

has positive upper density. Then F is a coboundary $\operatorname{mod}\lambda$.

PROOF. We substitute $f = 1$ in condition (ii) and use the fact that $F_n = U^n 1$. □

Now let T be just a rotation of $\mathbf{T}$ by an irrational angle θ. Veech [1] proved the following result (unfortunately, this is the only application of Theorem 1 known to the author).

THEOREM 2. *The function $F\colon \mathbf{T}\to\mathbf{T}$ is a coboundary* (mod 1) *iff*

$$(*)\qquad \int_{\mathbf{T}} F_n(x)\,d\mu(x) \to 1, \quad \textit{as } ||n\theta|| \to 0.$$

PROOF. We are going to derive this theorem from Theorem 1. It is easy to show that if F is a coboundary then (iii) holds. The other implication can be proved as follows. From $(*)$ we know that for some $\delta > 0$ $||n\theta|| < \delta$ implies $|\int F_n(x)\,d\mu(x)| > 1/2$. The set of n such that $||n\theta|| < \delta$ has positive upper density. Therefore the assumptions of Theorem 1 are satisfied and F is a coboundary $\operatorname{mod}\lambda$. It is easy to see that $(*)$ forces $\lambda = 1$. □

Theorem 1 generalizes without many changes to the actions of $\mathbf{Z}^n$ on X. We would like to conjecture that this theorem generalizes to the actions of amenable groups.

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