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COMPACTIFICATIONS OF COUNTABLE-DIMENSIONAL AND STRONGLY COUNTABLE-DIMENSIONAL SPACES

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ABSTRACT. Simple proofs of theorems on existence of compactifications of countable-dimensional and strongly countable-dimensional spaces are given.

In this note we present simple proofs of the following two theorems (for the terminology see [**2** and **3**]):

THEOREM 1. *Every countable-dimensional completely metrizable separable space has a countable-dimensional metrizable compactification.*

THEOREM 2. *Every strongly countable-dimensional completely metrizable separable space has a strongly countable-dimensional metrizable compactification.*

Let us recall that a completely regular space X is (strongly) countable-dimensional if X can be represented as a union of a sequence $X_1, X_2, \dots$ of (closed) subspaces with covering dimension $\dim X_i < \infty$ (see [**2**] and [**3**]).

Theorem 1 was announced by Hurewicz in [**5**] and proved by Lelek in [**8**]. Theorem 2 was established by Schurle in [**9**].

Theorem 1 follows immediately from the Lemma we state below. The original proof of the Lemma sketched in [**8**] was based on the technique of Kuratowski's κ-mappings to infinite polyhedra; cf. [**7**, §28, IX]; a short and elegant proof can be found in [**4**], it uses, however, some deep facts from infinite-dimensional topology. We give here an elementary proof.

LEMMA. *For every completely metrizable separable space X there exists an embedding $h: X \to Z$ into a compact metrizable space Z such that $Z \backslash h(X)$ is a countable union of finite-dimensional compact spaces.*

PROOF. One can consider X as a subspace of a compact metrizable space Y, and X is the intersection of countably many open subsets $U_1 \supset U_2 \supset \cdots$ of Y (see [**2**, Theorem 4.3.24]). Let ρ be an arbitrary metric on the space Y that is bounded by 1. Since ρ is totally bounded, for $i = 1, 2, \dots$ the set U_i can be represented as the union of finitely many open sets $U_{ij} \subset Y$, where $j \in J_i$, with diameter less than $1/i$; let $f_{ij}: Y \to I$ be the continuous function defined by $f_{ij}(y) = \rho(y, Y \backslash U_{ij})$ for $i = 1, 2, \dots$ and $j \in J_i$. Denote by f the diagonal of the functions f_{ij}; clearly, $f: Y \to I^{\aleph_0}$ and the restriction $h = f|X$ separates points and closed sets, and thus is an embedding of X into $Z = \overline{f(X)}$. As one easily sees, $Y \backslash X = f^{-1}(K_\omega)$, where K_ω is the subspace of $I^{\aleph_0}$ consisting of all points that have only finitely

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many nonzero coordinates; obviously, K_ω is the union of countably many finite-dimensional cubes. Now, $Z\backslash h(X) \subset K_\omega$, and being an F_σ-set in K_ω is a countable union of finite-dimensional compact spaces. □

In our proof of Theorem 2 we use a very special case of the separation theorem for Borel sets (see [**7**, §30, VII]), viz. the fact that for every pair A, B of disjoint G_δ-sets in a metrizable space M there exists a set C which is both an F_σ and a G_δ-set such that $A \subset C \subset M\backslash B$. This can be easily verified: if $A = \bigcap_{i=1}^\infty G_i$ and $B = \bigcap_{i=1}^\infty H_i$, where the sets $M = G_1 \supset G_2 \supset \cdots$ and $M = H_1 \supset H_2 \supset \cdots$ are open in M, the F_σ-set $C = \bigcup_{i=1}^\infty (G_i \backslash H_i)$ has the required properties, since $M\backslash C = \bigcup_{i=1}^\infty (H_i \backslash G_{i+1})$ is an F_σ-set, too.

PROOF OF THEOREM 2. Consider a strongly countable-dimensional completely metrizable space X; let $X = \bigcup_{i=1}^\infty F_i$, where F_i are finite-dimensional closed subsets of X. By a theorem of Hurewicz [**6**]; for a proof, see [**7**, §45, VII]), there exists a metrizable compactification M of the space X such that for $i = 1, 2, \ldots$ we have $\dim \overline{F}_i = \dim F_i$, where the closure is taken in M. The sets $A = X$ and $B = M\backslash \bigcup_{i=1}^\infty \overline{F}_i$ are disjoint G_δ-sets in M, so that there exists a set Y which is both an F_σ and a G_δ-set such that $X \subset Y \subset \bigcup_{i=1}^\infty \overline{F}_i$. The space Y is a countable union of finite-dimensional compact spaces; it is also completely metrizable, so that by the Lemma it has a metrizable compactification Z obtained by adjoining to Y countably many finite-dimensional compact spaces. Obviously, Z is a strongly countable-dimensional compactification of the space X. □

To conclude, let us observe that our proof of the Lemma yields a more general result:

PROPOSITION 1. *For every completely regular space X that has the following property:*

(*) *X is the intersection of countably many open subsets $U_1 \supset U_2 \supset \cdots$ of a compact space Y and for $i = 1, 2, \ldots$ the set U_i can be represented as the union of a point-finite family $\{U_{is}\}_{s \in S_i}$ of functionally open subsets of Y in such a way that for every pair x_1, x_2 of distinct points of X there exists an i and an $s \in S_i$ such that the set U_{is} contains exactly one of the points x_1, x_2,*

there exists an embedding $h: X \to Z$ into a compact space Z such that $Z\backslash h(X)$ is a countable union of finite-dimensional compact spaces.

PROOF. Let $f_{is}: Y \to I$ be a continuous function such that $f_{is}^{-1}(0) = Y\backslash U_{is}$ for $i = 1, 2, \ldots$ and $s \in S_i$. The diagonal f of the functions f_{is} is a continuous mapping of Y into a Tychonoff cube I^m, the restriction $h = f|X$ is a one-to-one mapping, and $Y\backslash X = f^{-1}(K)$, where K is the subspace of I^m consisting of all points that have only finitely many nonzero coordinates. Since h is the restriction of the perfect mapping f to the set $X = f^{-1}(I^m\backslash K)$, it is perfect itself (see [**2**, Proposition 3.7.4]), and being one-to-one is an embedding of X into $Z = \overline{f(X)}$. The remainder $Z\backslash h(X)$ is an F_σ-set in Z (see [**2**, Theorem 3.9.1]) and is contained in K, so that to conclude the proof it suffices to show that K is the union of countably many finite-dimensional compact spaces.

Clearly, $K = \bigcup_{n=0}^\infty K_n$, where K_n consists of all points in K that have at most n nonzero coordinates. The subspaces K_n being compact, it suffices to show that

$\dim K_n \leq n$ for $n = 0, 1, \ldots$. This is done by induction. Since K_0 consists of one point, $\dim K_0 = 0$. Assume that $\dim K_n \leq n$. One easily checks that each point in $K_{n+1} \setminus K_n$ has a neighborhood in K_{n+1} homeomorphic to the cube $(a, 1]^{n+1}$, where $0 < a < 1$; thus, for each closed set $F \subset K_{n+1}$ disjoint from K_n we have $\dim F \leq n + 1$ by the sum theorem, so that $\dim K_{n+1} \leq n + 1$ (see [**2**, Theorem 7.2.1 and Problem 7.4.17]). □

One easily checks that the class of completely regular spaces that have property $(*)$ is hereditary with respect to closed subspaces and countably multiplicative. Since every completely metrizable space is for some $m \geq \aleph_0$ homeomorphic to a closed subspace of $[J(m)]^{\aleph_0}$, where $J(m)$ is the hedgehog space of spininess m (see [**2**, Exercise 4.4B]), to show that all completely metrizable spaces have property $(*)$, it suffices to find an appropriate compact space containing $J(m)$. Such a space is, however, easily defined (cf. the proof of Theorem 14 in [**1**]): if S is the set of cardinality m used in the construction of $J(m)$, as described in [**2**, Example 4.1.5], then the quotient space $Y = (I \times \omega S)/(\{0\} \times \omega S)$, where ωS is the one-point compactification of the discrete space S, has all the required properties.

Thus, one obtains the following proposition that answers a question in [**3**] (the first part of Problem 5.4):

PROPOSITION 2. *Every countable-dimensional completely metrizable space has a countable-dimensional compactification.* □

REFERENCES

1. A. V. Arhangel'skiĭ, *Bicompacta and unions of countable families of metrizable spaces*, Dokl. Akad. Nauk SSSR **232** (1977), 989–992; English transl., Soviet Math. Dokl. **18** (1977), 165–169.
2. R. Engelking, *General topology*, PWN, Warsaw, 1977.
3. R. Engelking and E. Pol, *Countable-dimensional spaces*, Dissertationes Math. **216** (1983).
4. D. J. Garity and R. M. Schori, *Infinite-dimensional dimension theory*, Topology Proc. **10** (1985), 59–74.
5. W. Hurewicz, *Über unendlich-dimensionale Punktmengen*, Proc. Akad. Amsterdam **31** (1928), 916–922.
6. ———, *Über Einbettung separabler Räume in gleichdimensionale kompakte Räume*, Monatsh. Math. Phys. **37** (1930), 199–208.
7. K. Kuratowski, *Topology*, vols. I, II, Academic Press, New York, 1966 and 1968.
8. A. Lelek, *On the dimension of remainders in compact extensions*, Dokl. Akad. Nauk SSSR **160** (1965), 534–537; English transl., Soviet Math. Dokl. **6** (1965), 136–140.
9. A. Schurle, *Compactification of strongly countable-dimensional spaces*, Trans. Amer. Math. Soc. **136** (1969), 25–32.

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