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CRITICAL VALUES OF FREDHOLM MAPS

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ABSTRACT. For a C^r Fredholm map $A: X \to Y$, let $T_k(A)$ be the set of $x \in X$ such that codim Range $DA(x) \geq k$. The homotopy of $Y - A(T_k(A))$ is related to that of Y, examples are given, and a factorization result is proved.

THEOREM 1. *Let X and Y be C^r manifolds modeled on (real) Banach spaces, X separable and Y metrizable, let $A: X \to Y$ be a C^r Fredholm map of index ν, and let $T_k(A)$ be the set of $x \in X$ such that* codim Range $DA(x) \geq k$ $(k = 0, 1, \ldots)$. *Then the homomorphism*

$$i_*: \pi_m(Y - A(T_k(A)),\ y_0) \to \pi_m(Y, y_0)$$

on the mth homotopy groups $(m = 0, 1, \ldots)$ *induced by inclusion i is*

(a) *an isomorphism (onto) for $m + 2 \leq k$ and $r = 1 + \max(\nu + m + 1, 0)$, and is*

(b) *onto for $m + 1 \leq k$ and $r = 1 + \max(\nu + m, 0)$.*

Differentiable (C^r) Fredholm maps (Definition 2), defined by Smale [**Sm**], are useful in the study of certain nonlinear elliptic boundary value problems. Assuming $r > \max(\nu, 0)$ he proved the Smale-Sard Theorem [**Sm**, p. 862, (1.3)]: the set of critical (or singular) values $A(T_1(A))$ is meager in Y; for non-Fredholm maps this conclusion need not be true, even if $r = \infty$ [**K**]. Since a Fredholm map is locally proper [**Sm**, p. 862, (1.6)], the Smale-Sard Theorem is a special case of Theorem 1, viz. case (b) with $m = 0$ and $k = 1$.

The analog of Theorem 1 for singular homology with integer coefficients follows from the Whitehead Theorem [**Sp**, p. 399, Theorem 9]. For the finite dimensional case Theorem 1 is given in [**C-4**, p. 1035, Theorem 1]. The differential hypotheses are best possible (Example 5) if $k = m + 2$ in case (a) and $k = m + 1$ in case (b). By [**CT-2**, (1.5)b)] (and local properness [**Sm**, p. 862, (1.6)]) Theorem 1, case (a) with $m = 0$ and $k = 2$ is [**Mi**, p. 291, Theorem A], except that for $\nu < 0$ our hypothesis is C^1, while she requires C^2. In case $m = 0$ isomorphism in Theorem 1 means bijection.

DEFINITION 2. If E and F are Banach spaces, a bounded linear transformation $\Phi: E \to F$ is *Fredholm* if (i) $\dim\ker\Phi < \infty$, (ii) Range Φ is closed in F, and (iii) codim Range $\Phi < \infty$; *index* Φ is defined to be $\dim\ker\Phi$ − codim Range Φ. A C^1 map $A: X \to Y$ on C^1 Banach manifolds is *Fredholm* if $DA(x)$ is Fredholm for each $x \in X$. If X is connected, then index $DA(x)$ is independent of X and is called *index* A; if X is not connected, we assume that index $DA(x)$ is independent

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of component of X. The *singular set*, or *critical set*, SA is $T_1(A)$, the set of $x \in X$ for which $DA(x)$ is not surjective.

LEMMA 3. *Let (Δ, Γ) be a finite polyhedral pair such that $\Delta - \Gamma$ is a t-manifold (without boundary), let Y be a C^r $(r = 1, 2, \dots)$ manifold with metric d modeled on a Banach space E, let $\gamma \colon \Delta \to Y$ be C^0, and let $\rho \colon \Delta \to [0, \infty)$ be C^0 and 0 precisely on Γ. Then there exists a C^0 map $\delta \colon \Delta \to Y$ such that $\delta \mid (\Delta - \Gamma)$ is C^r and $d(\gamma(x), \delta(x)) \le \rho(x)$ (thus $\delta \mid \Gamma = \gamma \mid \Gamma$).*

In the following proof int N will refer to $(\mathrm{int}_\Delta N) \cap (\Delta - \Gamma) = \mathrm{int}_{\Delta - \Gamma} N$, while bdy N will refer to $\mathrm{Cl}_\Delta N - \mathrm{int}\, N$.

PROOF. We may suppose that $\Gamma = \mathrm{bdy}(\Delta - \Gamma)$. There are a finite number of charts $\phi_i \colon U_i \to E$ $(i = 1, 2, \dots, n)$ such that $\gamma(\Delta) \subset \bigcup_i U_i$. We may suppose that $\phi_i(U_i)$ is convex. Let $\eta > 0$ be the Lebesgue number of the open cover $\gamma^{-1}(U_i)$ of Δ, give Δ a subdivision with mesh less than $\eta/2$, and let Δ_i be the union of the closed t-simplices $\sigma \subset \gamma^{-1}(U_i)$; then Δ_i is a finite polyhedron, $\gamma(\Delta_i) \subset U_i$, and $\Delta = \bigcup_i \mathrm{int}_\Delta \Delta_i$. Let $\varepsilon > 0$ be the minimum of $d(\gamma(\Delta_k), Y - U_k)$ $(k = 1, 2, \dots, n)$, and define $\rho_i \colon \Delta \to [0, \infty)$ to be a C^0 map such that $\rho_i(x) > 0$ if and only if $x \in \mathrm{int}\, \Delta_i$ and $\rho_i(x) \le \min(\varepsilon, \rho(x))/n$.

It suffices to define inductively C^0 maps $\delta_i \colon \Delta \to Y$ $(i = 0, 1, \dots, n)$ such that $\delta_0 = \gamma$, δ_i is C^r on $\Omega_i = \bigcup_{k \le i} \mathrm{int}\, \Delta_k$ and $d(\delta_i(x), \delta_{i-1}(x)) \le \rho_i(x)$ for $i > 0$ and all $x \in \Delta$. Then δ_n will be the desired function δ.

Suppose that $\delta_0, \delta_1, \dots, \delta_{i-1}$ have been defined $(i > 0)$; we now define δ_i. There exists a C^0 map $\tau_i \colon \Delta_i \to [0, \infty)$ such that $\tau_i(x) = 0$ if and only if $x \in \mathrm{bdy}\, \Delta_i$ and

(1) if $x \in \Delta_i$ and $y \in U_i$ with $\|\phi_i(\delta_{i-1}(x)) - \phi_i(y)\| \le \tau_i(x)$, then $d(\delta_{i-1}(x), y) \le \rho_i(x)$. For each $x \in \mathrm{int}\, \Delta_i$, choose an open neighborhood V of x such that

(2) diam $(\overline{V}) < (1/2) d(\overline{V}, \mathrm{bdy}\, \Delta_i)$ (thus $\overline{V} \subset \mathrm{int}\, \Delta_i$) and,

(3) if $y, z \in \overline{V}$, then $\|\phi_i(\delta_{i-1}(y)) - \phi_i(\delta_{i-1}(z))\| < \min\{\tau_i(w) \colon w \in \overline{V}\}$. The V constitute an open cover of $\mathrm{int}\, \Delta_i$; let W_j $(j = 1, 2, \dots)$ be a countable locally finite refinement, and let $W_0 = \Omega_{i-1} \cap \mathrm{int}\, \Delta_i$. Let ψ_j be a C^∞ partition of unity subordinate to the cover W_j $(j = 0, 1, \dots)$; we may suppose that $\phi \ne W_j \not\subset W_0$ for $j > 0$, so (by 2))

(4) each point of $\Omega_{i-1} \cap \mathrm{bdy}\, \Delta_i$ has an open neighborhood disjoint from $\bigcup_{j > 0} \overline{W}_j$. Define a C^0 map $\eta_j \colon w_j \to U_i$ as follows: $\eta_0 = \delta_{i-1} | W_0$, and for $j > 0$, pick $x_j \in W_j$, and let $\eta_j(x) \equiv \delta_{i-1}(x_j)$. For $x \in \mathrm{int}\, \Delta_i$ let

$$\delta_i(x) = \phi_i^{-1} \left(\sum_{j \ge 0} \psi_j(x) \phi_i(\eta_j(x)) \right),$$

and otherwise let $\delta_i(x) = \delta_{i-1}(x)$. It follows from (3) that for $x \in \mathrm{int}\, \Delta_i$,

$$\begin{aligned} &\|\phi_i(\delta_i(x)) - \phi_i(\delta_{i-1}(x))\| \\ &\quad \le \sum_{j > 0} \psi_j(x) \|\phi_i(\delta_{i-1}(x_j)) - \phi_i(\delta_{i-1}(x))\| \\ &\quad \le \sum_{j > 0} \psi_j(x) \tau_i(x) \le \tau_i(x), \end{aligned}$$

and by (1) $d(\delta_i(x), \delta_{i-1}(x)) \leq \rho_i(x)$ for all $x \in \Delta$. From (4) each point of $\Omega_{i-1} \cap \mathrm{bdy}\Delta_i$ has an open neighborhood on which δ_i agrees with δ_{i-1}, so (by the inductive hypothesis) δ_i is C^r on Ω_i, and δ_i has the desired properties.

4. PROOF OF THEOREM 1. According to [**Q**, p. 215, Theorem 4] and [**BS**, p. 6, Theorem 1], if M is a separable finite dimensional manifold and $r > (\dim M) + \nu$, then the maps transversal to A are dense in $C^r(M, Y)$ in the fine C^r topology [**Mu**, p. 29 and p. 32]. (In [**Q**, p. 215] the calculations show that $\mathrm{index}(\pi_A \circ h) = \dim M + \mathrm{index}\, F$, not $\dim M - \mathrm{index}\, F$ as stated, and the corresponding change must be made in [**Q**, p. 215, Theorem 4].) Thus (1) if $\gamma\colon \Delta \to Y$ is the C^0 map of Lemma 3, we may suppose that the approximation δ is C^r transversal to the map A at each point of $\Delta - \Gamma$.

Let (Δ, Γ) be a finite polyhedral pair, and let $\gamma\colon \Delta \to Y$ and $\sigma\colon \Delta \to [0, \infty)$ be C^0 maps with $\sigma(x) = 0$ if and only if $x \in \Gamma$. The Banach manifold Y is an ANR [**E**, p. 766 or **H-2**, p. 95; p. 98, Theorem 8.1; and p. 96, Corollary 6.4 and Corollary 6.5]. From [**H-2**, p. 112, Theorem 1.2] there exists a C^0 map $\rho\colon \Delta \to [0, \infty)$ with $\rho(x) = 0$ if and only if $x \in \Gamma$ such that, if $\delta\colon \Delta \to Y$ is C^0 and $d(\gamma(x), \delta(x)) \leq \rho(x)$, then there is a C^0 homotopy $h_t\colon \Delta \to [0, \infty)$ satisfying $h_0(x) = \gamma(x)$, $h_1(x) = \delta(x)$, and $d(\gamma(x), h_t(x)) \leq \sigma(x)$.

Let $\gamma = (S^m, s_0) \to (Y, y_0)$, let $(\Delta, \Gamma) = (S^m, s_0)$, and let ρ be given above (for any σ). Let δ be given by (1) and Lemma 3 for this ρ if $r > \max(m + \nu, 0)$; if $m + 1 \leq k$, then $\gamma(S^m - s_0) \cap A(T_k(A)) = \varnothing$, so that γ represents an element of $\pi_m(Y - A(T_k(A)), y_0)$ as desired.

Let $\gamma\colon (D^{m+1}, S^m, s_0) \to (Y, Y - A(T_k(A)), y_0)$, let $(\Delta, \Gamma) = (D^{m+1}, S^m)$, let ρ be given for any σ, and let δ be given by (1) and Lemma 3 if $r > \max(m + 1 + \nu, 0)$; if $m + 2 \leq k$, then $\gamma(D^{m+1} - S^m) \cap A(T_k(A)) = \varnothing$, so that γ represents the trivial element of $\pi_m(Y - A(T_k(A)), y_0)$.

The proof shows more than is stated: any $\gamma\colon (S^m, s_0) \to (Y, y_0)$ with $y_0 \notin A(T_k(A))$ may be approximated by a homotopic $\delta\colon (S^m, s_0) \to (Y - A(T_k(A)), y_0)$, and if δ is homotopic to the constant y_0 in (Y, y_0), that homotopy may be approximated by a homotopy in $(Y - A(T_k(A)), y_0)$.

EXAMPLE 5. *The differentiability hypotheses in Theorem* 1 *are sharp for all integers* $m \geq 0$, ν *and* (a) $k = m + 2$ [*resp.*, (*b*) $k = m + 1$], *i.e.*, *injectivity in* (a) *and surjectivity in* (b) *are false if* C^r *is replaced by* C^{r-1}.

Let $P(s)$ be the statement: if $u > s$ $(s, u = 1, 2, \dots)$, then there exists a C^{u-s} map $h\colon \mathbb{R}^u \to \mathbb{R}^s$ such that $h(R_0(h)) = \mathbb{R}^s$, where $R_0(h) = T_s(h)$ is the set of points in $\mathbb{R}^u$ at which Dh has rank 0. According to [**Whi**], $P(1)$ is true (see also [**R**]). The authors have not constructed examples to show that $P(s)$ is true for $s > 1$, although it seems plausible that a generalization of Whitney's construction might be successful. We prove below:

If $P(s)$ is true for a given s then for every integer $m \geq 0$, ν, and (a) $k = s + m + 1$ [resp., (b) $k = s + m$], the differentiability hypothesis in Theorem 1 is sharp. Thus Example 5 results, and, if $P(s)$ is true for all positive integers s, then the differentiability hypothesis in Theorem 1 is sharp for all m, ν and k (a) $m + 2 \leq k$ [resp., (b) $m + 1 \leq k$].

PROOF. The differentiability hypothesis is (a) $r = 1 + \max(\nu + m + 1, 0)$ [resp., (b) $r = 1 + \max(\nu + m, 0)$]. If $r = 1$, it is certainly sharp, so we may suppose that (a) $\nu + m \geq 0$ [resp., (b) $\nu + m \geq 1$]. Given any integers $m \geq 0$, ν, and

(a) $k \geq m+2$ [resp., (b) $k \geq m+1$], let (a) $s = k-m-1$ [resp., (b) $s = k-m$], let $u = k+\nu$, let $h\colon \mathbb{R}^u \to \mathbb{R}^s$ be the (a) $C^{\nu+m+1}$ [resp., (b) $C^{\nu+m}$] map given by $P(s)$, and define $\alpha\colon \mathbb{R}^s \to \mathbb{R}^k = \mathbb{R}^s \times \mathbb{R}^{k-s}$ by $\alpha(x) = (x,0)$. Let $A = \alpha h$, so that $A(T_k(A)) = \alpha(\mathbb{R}^s)$; in case (a) $\pi_m(\mathbb{R}^k - A(T_k(A)) \neq 0$ as desired.

In case (b) let $\Gamma = \{x \in \mathbb{R}^{m+1}\colon 1 < \|x\| < 2\}$ and let $V = \mathbb{R}^{s-1} \times \Gamma \subset \mathbb{R}^k$. Then V and $V - \alpha(\mathbb{R}^s)$ have the homotopy types of S^m and S^{m-1}, respectively ($S^{-1} = \varnothing$). Let $U = A^{-1}(V)$, and let $B\colon U \to V$ be the restriction of A; then $B(T_k(B)) = \alpha(\mathbb{R}^s) \cap V$, and B is the desired example for (b). [Actually example B for (b) is also an example for (a): given an $m \neq 0$ for (b), $m-1$ is an m for (a).]

These examples $A, B\colon \mathbb{R}^n \to \mathbb{R}^p$ have minimal p $(p = k)$, and other examples are $A \times \mathrm{id}_E$ and $B \times \mathrm{id}_E$, where id_E is the identity on a Banach space E. Some related examples are given in [**C-3**, p. 421, (3.3)] for case (a), and in [**Sa**, p. 890] for case (b) with $m = 0$.

PROPOSITION 6 [**C-4**, p. 1037, Proposition 4]. *If $A\colon M^n \to N^p$ is a C^r map on C^r manifolds with $r \geq \max(n-p+k, 1)$, then* $\dim(A(T_k(f))) \leq p-k$.

EXAMPLE 7. *For all positive integers n, p, and k $(k \leq p)$, the differential hypotheses in Proposition* 6 *are sharp.*

PROOF. If $r = 1$, the hypothesis is sharp, so we may suppose that $n-p+k \geq 2$. Let $u = n-p+k$, let $h\colon \mathbb{R}^u \to \mathbb{R}$ be the $C^{n-p+k-1}$ map of $P(1)$ in Example 5, define $g\colon \mathbb{R}^n = \mathbb{R}^u \times \mathbb{R}^{p-k} \to \mathbb{R} \times \mathbb{R}^{p-k}$ by $g(x,y) = (h(x), y)$, define $\alpha\colon \mathbb{R}^{1+p-k} \to \mathbb{R}^p = \mathbb{R}^{1+p-k} \times \mathbb{R}^{k-1}$ by $\alpha(x) = (x,0)$, and let $A = \alpha g$.

COROLLARY 8. *Let $A\colon X \to Y$ be a C^r proper Fredholm map of index ν, where $r \geq \max(\nu+3, 1)$ and X and Y are C^r connected Banach manifolds with X separable. Suppose that* (a) *for every nonempty connected open set $U \subset X$, $U - SA$ is nonempty and connected, and* (b) *for every $x \in X$,* codim Range $DA(x) \neq 1$.

(I) *Then $\nu \geq 0$ and there are s $(s = 1, 2, \ldots)$ and a compact connected $C^r\nu$-manifold F^ν without boundary such that,*

(i) *for each $y \in Y - A(SA)$, $A^{-1}(y)$ is homeomorphic to s disjoint copies of F^ν, and*

(ii) *for $y \in A(SA)$, $A^{-1}(y)$ has at most s components.*

(II) *If, in addition,* (c) codim $DA(x) \neq 2$ *for every $x \in X$, then there is a factorization $A = \Psi\Phi$, where*

(i) *Z is a C^r connected Banach manifold,*

(ii) *$\Psi\colon Z \to Y$ is a C^r s-to-1 $(s = 1, 2, \ldots)$ diffeo-covering map,*

(iii) *$\Phi\colon X \to Z$ is a C^r Fredholm map of index ν and, for every $z \in Z$, the set of solutions x of $\Phi(x) = z$ is a nonempty compact connected set. In particular, if $z \notin \Phi(S\Phi)$ and $\nu = 0$, then there is precisely one solution.*

(III) *The factorization is unique in the following sense: If $A = \bar{\Psi}\bar{\Phi}$ is another factorization with intermediate manifold $\bar{Z}$, then there is a C^r diffeomorphism $\alpha\colon Z \to \bar{Z}$ such that $\bar{\Phi} = \alpha\Phi$ and $\bar{\Psi} = \Psi\alpha^{-1}$.*

PROOF. Let V be any connected open set in Y. By hypothesis (b) the singular set SA is $T_2(A)$, the set of $x \in X$ such that codim Range $DA(x) \geq 2$, and by Theorem 1, $V - A(SA)$ is connected. Since A is proper, $A(SA)$ is closed in Y and

$$(1) \qquad B_V = A \mid [A^{-1}(V - A(SA))]\colon A^{-1}(V - A(SA)) \to V - A(SA)$$

is a C^0 bundle map [**CK**, argument of p. 151] whose fiber has s components ($s = 0, 1, \dots$). From hypothesis (a) and Theorem 1 neither SA nor $A^{-1}(A(SA))$ can locally separate $A^{-1}(V)$ at any point, so that (2) for each component U_i of $A^{-1}(V)$, $U_i - A^{-1}(A(SA)) \neq \emptyset$ and is a component of $A^{-1}(V - A(SA))$. Since A is proper, $A(U_i)$ is closed in V and since (by (1) and (2)) $V - A(SA) \subset A(U_i)$, (3) $A(U_i) = V$. Thus (4) $\nu \geq 0$, s is independent of the choice of V (by (1)), $s > 0$, and (5) $A^{-1}(V)$ has at most s components. If $A^{-1}(y)$ has at least t components, then there is an open ball neighborhood V of y such that $A^{-1}(V)$ has at least t components. By (5), $t \leq s$, so that (6) for every $y \in Y$, $A^{-1}(y)$ has at most s components, and if $y \notin A(SA)$, $A^{-1}(y)$ has exactly s components by (1).

From (2) domain (B_Y) is connected, and (7) B_Y has a factorization $\varsigma\eta$, where ς is a C^0 bundle map with connected fiber and ς is a C^0 s-to-1 covering map. (This is the monotone-light factorization [**Why**, pp. 141–143] of the proper map A.) Conclusion (I) follows from (4), (6), and (7).

Now assume hypothesis (II)(c) and that V is simply connected, so that (Theorem 1) $V - A(SA)$ is simply connected. For each component U_i of $A^{-1}(V)$,

$$B_V \mid [U_i - A^{-1}(A(SA))] \colon U_i - A^{-1}(A(SA)) \to V - A(SA)$$

is a C^0 bundle map with connected domain by (2), and from the homotopy sequence of a fibering [**H-1**, p. 152] its fiber is connected. From (3) $A^{-1}(V)$ and $A^{-1}(y)$ for $y \in V$ each have precisely s components. The factorization in the C^0 category is immediate, and Z is given the C^r structure induced from Y by Ψ, so that Φ is C^r and Ψ is a C^r local diffeomorphism and s-to-1 covering map. Conclusion (II) results and conclusion (III) (uniqueness) is proved as in [**C-2**, p. 385, (3.6)].

REMARK 9. Other factorization results, in finite dimensions, are given in [**C-2**] and [**CT-1**]. Hypothesis (a) is required [**C-1**, p. 707, Examples 11 and 12].

COROLLARY 10. *Let $A \colon X \to Y$ be a C^r proper Fredholm map of index ν, where $r \geq \max(\nu + 3, 1)$, X is a C^r connected separable Banach manifold and Y is a Banach space. Suppose that* (a) *for every nonempty connected open set $U \subset X$, $U - SA$ is nonempty and connected and* (b) *for every $x \in X$,* codim Range $DA(x)$ *is neither* 1 *nor* 2. *Then* (i) $\nu \geq 0$; (ii) *for each $y \in Y$, the set of solutions x of $A(x) = y$ is a nonempty compact connected set*; (iii) *in particular, if $\nu = 0$ and $y \notin A(SA)$, then there is precisely one solution.*

PROOF. Since Y is simply connected, Ψ in Corollary 8 is a diffeomorphism [**Ma**, p. 159, Theorem 6.6 or p. 160, Exercise 6.1].

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