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RELATION BETWEEN GROWTH AND REGULARITY OF SOLUTIONS OF HYPOELLIPTIC EQUATIONS

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ABSTRACT. For a class of linear partial differential equations with variable coefficients, it is shown that the Gevrey regularity of solutions depends on their growth at infinity.

Let $P(D)$ be a partial differential operator with constant coefficients. If $P(D)$ is hypoelliptic, then all distributions u in R^n, solutions of the equation

$$P(D)u = 0, \tag{1}$$

are C^∞-functions which belong to a Gevrey class $\Gamma^d(R^n)$, where $d = (d_1, \ldots, d_n)$ and $d_j \geq 1$, $j = 1, \ldots, n$. This means that for every compact set $K \subset R^n$ there is a constant $C > 0$ such that

$$|D^\alpha u(x)| \leq C^{|\alpha|+1} \alpha_1^{\alpha_1 d_1} \alpha_2^{\alpha_2 d_2} \cdots \alpha_n^{\alpha_n d_n}, \qquad x \in K \tag{2}$$

for every multi-index α.

V. V. Grusin [**2**] has shown that, for a given solution u of equation (1), the Gevrey class $\Gamma^d(R^n)$ depends not only on the differential operator $P(D)$ but also on the growth of u at infinity. In fact, the numbers d_j, $j = 1, \ldots, n$, can be lowered and condition (2) can be replaced by a global condition in R^n, if one considers solutions of finite exponential order of growth in R^n.

The aim of this paper is to extend Grusin's investigations to a class of partial differential operators with variable coefficients.

1. The case of operators with constant coefficients. We recall briefly some of the results obtained in [**2**].

Let $P(D)$ be a differential operator with constant coefficients and $P(\xi)$ the corresponding polynomial. We denote by $\mathscr{N}_k$ the set of all $\varsigma = (\varsigma_1, \ldots, \varsigma_n) \in \mathbf{C}^n$ such that $P(\varsigma) = 0$ and $\operatorname{Im} \varsigma_j = 0$ for $j \neq k$. If $P(D)$ is hypoelliptic, then there are rational numbers $d_j^k > 0$ and constants $C_j^k > 0$, $j, k = 1, \ldots, n$, with the following properties:

(h_1) If $\varsigma = \xi + i\eta$, where $\xi = (\xi_1, \ldots, \xi_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ are in R^n, then

$$|\xi_j| \leq C_j^k (1 + |\eta|)^{d_j^k} = C_j^k (1 + |\eta_k|)^{d_j^k}. \tag{3}$$

(h_2) d_j^k are the smallest numbers for which the inequalities (3) are valid with some constants C_j^k.

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We call the numbers d_j^k the exponents of hypoellipticity of the operator $P(D)$. We note that d_j^k corresponds to $1/\gamma_j^k$ in [**2**].

It is well known that every differential operator $P(D)$ with constant coefficients has a tempered fundamental solution, i.e. there exists a tempered distribution E such that

$$P(D)E = \delta \tag{4}$$

where δ is the Dirac measure. For a hypoelliptic operator $P(D)$, every tempered fundamental solution E has the following properties (see [**2**, Theorem 3.1]):

(e_1) If $d_j^k, k, j = 1, \ldots, n$, are exponents of hypoellipticity of $P(D)$ and $d_j = \max_{1 \leq k \leq n} d_j^k \geq 1$, then $E \in \Gamma^d(R^n \backslash \{0\})$, where $d = (d_1, \ldots, d_n)$.

(e_2) There exists an integer $l \geq 0$ such that

$$D^\alpha E(x) = O((1 + |x|)^l) \quad \text{as } |x| \to \infty, \tag{5}$$

for every multi-index α.

In particular, the property (e_1) implies that every distribution u, solution of equation (1) in R^n, belongs to $\Gamma^d(R^n)$ (see [**4**, Proposition 7.2, p. 413]).

We now state a version of the theorem of Grusin [**2**, Theorem 4.1]) which establishes the relation between growth and regularity of solutions of equation (1).

THEOREM 1. *Let $P(D)$ be a hypoelliptic differential operator and $d_j^k, j, k = 1, \ldots, n$, its exponents of hypoellipticity. Furthermore, let u be a solution of equation* (1) *which satisfies the growth condition*

$$|u(x)| \leq A \exp\left(a \sum_{k=1}^n |x_k|^{p_k}\right), \qquad x \in R^n, \tag{6}$$

where A and a are positive constants and $p_k > 1$, $k = 1, \ldots, n$. Then there exist constants $C > 0$ and $c \geq a$ such that

$$\left|\frac{\partial^m u(x)}{\partial x_j^m}\right| \leq AC^m \left(\sum_{k=1}^n m^{m d_j^k / q_k}\right) \exp\left(c \sum_{k=1}^n |x_k|^{p_k}\right), \qquad x \in R^n, \tag{7}$$

where $m = 1, 2, \ldots$, and $1/p_k + 1/q_k = 1$.

REMARK. Since $p_k > 1$, clearly $d_j^k / q_k < d_j^k$, $j, k = 1, \ldots, n$, which shows that the regularity is improved due to the restriction of the growth of u.

2. The case of operators with variable coefficients. We consider a differential operator of the form

$$P(x, D) = P_0(D) + \sum_{\nu=1}^r a_\nu(x) P_\nu(D) \tag{8}$$

where $P_\nu(D)$ are operators with constant coefficients and the functions a_ν satisfy certain regularity and growth conditions. Specifically, we make the following assumptions on the operators $P_\nu(D)$:

(c_1) The operator $P_0(D)$ is hypoelliptic. We denote by d_j^k, $j, k = 1, \ldots, n$, its exponents of hypoellipticity and assume that $d_j^k > 1$.

(c_2) For $j = 1, \dots, n$ and $\nu = 1, \dots, r$,

$$\int_{R^n} \frac{\widetilde{\xi_j P_\nu}(\xi)}{\tilde{P}_0(\xi)}\, d\xi < \infty, \tag{9}$$

where $\tilde{P}(\xi) = (\sum_\alpha |P^{(\alpha)}(\xi)|^2)^{1/2}$ and $P^{(\alpha)}(\xi) = D^\alpha P(\xi)$.

Note that, because of (c_1), $P_0(D)$ satisfies the conditions imposed on $P(D)$ in Theorem 1. Also, condition (c_2) implies that, for any $x_0 \in R^n$, $P(x_0, D)$ and $P_0(D)$ are equally strong. Hence $P(x_0, D)$ is hypoelliptic for every $x_0 \in R^n$, by (c_1) and Theorem 4.1.6 in [**3**]. Moreover, if $a_\nu, \nu = 1, \dots, r$, are C^∞-functions then every distribution u in R^n, solution of the equation

$$P(x, D)u = 0, \tag{10}$$

is a C^∞-function, by Theorem 7.4.1 in [**3**]. We wish to study solutions of equation (10) which satisfy the growth condition (6).

We make the following assumption on the functions a_ν:

(c_3) Let $\rho_j = \min_{1 \le k \le n} d_j^k/q_k$, $j = 1, \dots, n$, and let c be the constant in (7), when Theorem 1 is applied to the operator $P_0(D)$. Then there exist constants $B > 0$ and $b > c$ such that

$$|D^\alpha a_\nu(x)| \le B^{|\alpha|+1} \alpha_1^{\alpha_1 \rho_1} \cdots \alpha_n^{\alpha_n \rho_n} \exp\left(-b \sum_{k=1}^n |x_k|^{p_k}\right), \qquad x \in R^n \tag{11}$$

for all $\nu = 1, \dots, r$, and all multi-indices α.

REMARK. Since $d_j^k < 1$ and $1/p_k + 1/q_k = 1$, we have $1/p_k + d_j^k/q_k > 1, j, k = 1, \dots, n$, and therefore the family of functions satisfying condition (c_3) is not trivial, i.e. it contains functions that are not identically zero (see [**1**, Chapter IV, §8]).

Our main result is the following theorem.

THEOREM 2. *Let $P(x, D)$ be a differential operator of the form* (8), *where the operators $P_0(D)$ and $P_\nu(D)$ satisfy conditions* (c_1) *and* (c_2), *and the functions a_ν satisfy condition* (c_3). *If u is a solution of equation* (10) *which satisfies the growth condition* (6), *then there are positive constants A_1, C_1 and $c_1 > a$ such that*

$$\left|\frac{\partial^m u(x)}{\partial x_j^m}\right| \le A_1 C_1^m \left(\sum_{k=1}^n m^{m(1+d_j^k/q_k)}\right) \exp\left(c_1 \sum_{k=1}^n |x_k|^{p_k}\right), \qquad x \in R^n \tag{12}$$

where $m = 1, 2, \dots,$ and $1/p_k + 1/q_k = 1$.

We first prove two lemmas. In the first lemma, $\mathscr{S}$ denotes Schwartz's space of rapidly decreasing C^∞-functions.

LEMMA 1. *Let $P(D)$ and $Q(D)$ be differential operators with constant coefficients and suppose that $P(D)$ is hypoelliptic. If*

$$\int_{R^n} \frac{\tilde{Q}(\xi)}{\tilde{P}(\xi)}\, d\xi < \infty, \tag{13}$$

and if E is a fundamental solution for $P(D)$, then there are constants M and N such that

$$|(Q(D)E * \phi)(x)| \le M(1 + |x|)^N \int_{R^n} (1 + |y|)^N |\phi(y)|\, dy, \qquad x \in R^n, \tag{14}$$

where "$*$" denotes the convolution and $\phi \in \mathscr{S}$.

PROOF. Since $P(D)$ is hypoelliptic, there are positive constants a, c and C such that

$$|\xi|^c \leq C|P(\xi)| \quad \text{for } \xi \in G_a = \{\eta \in R^n; |\eta| \geq a\}.$$

Consider the function

$$\Phi(\xi) = \begin{cases} \frac{1}{P(-\xi)} & \text{for } \xi \in G_a, \\ 0 & \text{otherwise.} \end{cases}$$

If $\hat{E}$ is the Fourier transform of E, then $P\hat{E} = 1$. In particular, $\hat{E}(\xi) = 1/P(\xi)$ for $\xi \in G_a$. It follows that $H = (\hat{E})^{\check{}} - \Phi$ is a distribution with compact support and we have

$$E = \hat{H} + \hat{\Phi}; \tag{15}$$

$(\hat{E})^{\check{}}$ is defined by $(\hat{E})^{\check{}}(\phi) = \hat{E}(\check{\phi})$, where $\check{\phi}(\xi) = \phi(-\xi)$.

By the Paley-Wiener-Schwartz theorem, $\hat{H}$ is an entire function of exponential type and there are positive constants M_1 and N such that

$$|Q(D)\hat{H}(x)| \leq M_1(1 + |x|)^N, \qquad x \in R^n.$$

We note that N does not depend on the operator $Q(D)$. It follows that, for any $\phi \in \mathscr{S}$, we have

$$\begin{aligned} |(Q(D)\hat{H} * \phi)(x)| &\leq M_1 \int_{R^n} (1 + |x - y|)^N |\phi(y)|\, dy \\ &\leq M_1(1 + |x|)^N \int_{R^n} (1 + |y|)^N |\phi(y)|\, dy, \qquad x \in R^n. \end{aligned} \tag{16}$$

Also,

$$\widehat{Q(D)\hat{\Phi} * \phi} = Q(\xi)\check{\Phi}(\xi)\hat{\phi}(\xi),$$

so that

$$(Q(D)\hat{\Phi} * \phi)(x) = \frac{1}{(2\pi)^n} \int_{R^n} e^{i\langle x, \xi\rangle} Q(\xi)\check{\Phi}(\xi)\hat{\phi}(\xi)\, d\xi.$$

Hence

$$|(Q(D)\hat{\Phi} * \phi)(x)| \leq \frac{1}{(2\pi)^n} \int_{G_a} \left|\frac{Q(\xi)}{P(\xi)}\right| |\hat{\phi}(\xi)|\, d\xi, \qquad x \in R^n.$$

Using again the hypoellipticity of $P(D)$, we can find a constant $C' > 0$ such that

$$\tilde{P}(\xi) \leq C'|P(\xi)|, \qquad \xi \in G_a.$$

Consequently,

$$\begin{aligned} |(Q(D)\hat{\Phi} * \phi)(x)| &\leq \frac{C'}{(2\pi)^n} \sup_{\xi \in R^n} |\hat{\phi}(\xi)| \int_{G_a} \frac{|Q(\xi)|}{\tilde{P}(\xi)}\, d\xi \\ &\leq M_2 \int_{R^n} |\phi(y)|\, dy, \qquad x \in R^n, \end{aligned} \tag{17}$$

where

$$M_2 = \frac{C'}{(2\pi)^n} \int_{R^n} \frac{\tilde{Q}(\xi)}{\tilde{P}(\xi)} d\xi.$$

If $M = M_1 + M_2$, we obtain from (15), (16) and (17) the estimate (14).

COROLLARY. *If $P(D)$, $Q(D)$ and E are as in Lemma 1 and*

$$\int_{R^n} \frac{\widetilde{\xi_j Q}(\xi)}{\tilde{P}(\xi)} d\xi < \infty, \tag{18}$$

for some j, then

$$\left| \left(Q(D)E * \frac{\partial \phi}{\partial x_j} \right)(x) \right| \leq M(1+|x|)^N \int_{R^n} (1+|y|)^N |\phi(y)|\, dy, \qquad x \in R^n.$$

For the proof it suffices to apply Lemma 1 to the operator $D_j Q(D)$, where $D_j = \partial/\partial x_j$.

REMARK. Lemma 1 remains valid, if we assume that ϕ is a continuous function rapidly decreasing at infinity, i.e. that $(1+|x|)^k \phi(x)$ is bounded in R^n, for every k.

LEMMA 2. *Let $P(x,D)$ be a differential operator of the form* (8), *where $P_0(D)$, $P_\nu(D)$ and a_ν, $\nu = 1, \dots, r$, satisfy conditions* (c_1), (c_2) *and* (c_3), *and let E_0 be a tempered fundamental solution for $P_0(D)$. Then there are operators $Q_\nu(D)$ and functions b_ν, $\nu = 1, \dots, s$, with the following property: If u is a solution of the equation* (10) *which satisfies condition* (6), *then*

$$v = u + \sum_{\nu=1}^{s} Q_\nu(D) E_0 * (b_\nu u) \tag{19}$$

is a solution of the equation

$$P_0(D)v = 0. \tag{20}$$

Each polynomial $Q_\nu(\xi)$ is a derivative of some order of a polynomial $P_\mu(\xi)$, $\mu \geq 1$, and each function $b_\nu(\xi)$ is proportional to a derivative of some order of a function $a_\mu(\xi)$.

PROOF. The polynomials $Q_\nu(\xi)$ and the functions $b_\nu(\xi)$ obviously satisfy the conditions (9) and (11), respectively. In particular, the products $b_\nu u$ decrease at infinity faster than any power of $|\xi|^{-1}$. Therefore the convolutions in (19) are well defined.

By assumption

$$P_0(D)u + \sum_{\nu=1}^{r} a_\nu P_\nu(D)u = P(x,D)u = 0.$$

To each term $a_\nu P_\nu(D)u$ we now apply the generalized Leibniz formula

$$a_\nu P_\nu(D)u = \sum_{\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} P_\nu^{(\alpha)}(D)(u D^\alpha a_\nu).$$

In this way we obtain the equation

$$P_0(D)u + \sum_{\nu=1}^{s} Q_\nu(D)(b_\nu u) = 0.$$

Since E_0 is a fundamental solution for $P_0(D)$, we have

$$Q_\nu(D)(b_\nu u) = P_0(D)[Q_\nu(D)E_0 * (b_\nu u)]$$

for each $\nu = 1, \ldots, s$. Hence

$$P_0(D)\left[u + \sum_{\nu=1}^{s} Q_\nu(D)E_0 * (b_\nu u)\right] = 0$$

which proves the lemma.

PROOF OF THEOREM 2. If u satisfies condition (6), then v satisfies the same condition with another constant $A_0 \geq A$. Since, by Lemma 2, v is a solution of equation (20), we may apply Theorem 1 to conclude that

$$u = -\sum_{\nu=1}^{s} Q_\nu(D)E_0 * (b_\nu u) + f,$$

where $Q_\nu(D)$ and b_ν are as in Lemma 2 and f is a C^∞-function which satisfies the estimates (7). In particular, condition (18) is valid for $j = 1, \ldots, n$, with $Q(\xi)$ replaced with $Q_\nu(\xi)$, $\nu = 1, \ldots, s$, and the functions b_ν satisfy condition (11) with the constants B and b. Thus, in view of the corollary from Lemma 1, we have

$$\begin{aligned}\left|\frac{\partial u(x)}{\partial x_j}\right| &\leq M \sum_{\nu=1}^{s}(1+|x|)^N \int_{R^n}(1+|y|)^N |b_\nu(y)u(y)|\,dy + \left|\frac{\partial f(x)}{\partial x_j}\right| \\ &\leq A_0 C_1 (1+\lambda s M)\exp\left(c\sum_{k=1}^{n}|x_k|^{p_k}\right), \qquad x \in R^n\end{aligned}$$

where $C_1 = \max\{B, C\}$ and

$$\begin{aligned}\lambda = \sup_{x \in R^n}&\left[(1+|x|)^N \exp\left(-c\sum_{k=1}^{n}|x_k|^{p_k}\right)\right] \\ &\cdot \int_{R^n}(1+|y|)^N \exp\left[-(b-c)\sum_{k=1}^{n}|x_k|^{p_k}\right] dy.\end{aligned}$$

Suppose now that

$$\left|\frac{\partial^l u(x)}{\partial x_j^l}\right| \leq A_0 C_1^l (l+\lambda s M)^l \left(\sum_{k=1}^{n} l^{l d_j^k/q_k}\right) \exp\left(c\sum_{k=1}^{n}|x_k|^{p_k}\right), \qquad x \in R^n$$

for $l = 1, \ldots, m$. Then

(21)

$$\begin{aligned}\left|\frac{\partial^{m+1} u(x)}{\partial x_j^{m+1}}\right| &\leq M\sum_{\nu=1}^{s}(1+|x|)^N \int_{R^n}(1+|y|)^N \left|\frac{\partial^m [b_\nu(y)u(y)]}{\partial y_j^m}\right| dy + \left|\frac{\partial^{m+1} f(x)}{\partial x_j^{m+1}}\right| \\ &\leq A_0 C_1^{m+1}\left\{\lambda s M \sum_{\nu=0}^{m}\binom{m}{l} l^{l\rho_j}(l+\lambda s M)^{m-l}\left(\sum_{k=1}^{n}(m-l)^{(m-l)d_j^k/q_k}\right)\right. \\ &\qquad\qquad \left. + \left(\sum_{k=1}^{n}(m+1)^{(m+1)d_j^k/q_k}\right)\right\} \\ &\quad \cdot \exp\left(c\sum_{k=1}^{n}|x_k|^{p_k}\right), \qquad x \in R^n.\end{aligned}$$

Since, by definition, $\rho_j = \min_{1\le k\le n} d_j^k/q_k$, we have

$$l^{l\rho_j}(m-l)^{(m-l)d_j^k/q_k} \le l^{ld_j^k/q_k}(m-l)^{(m-l)d_j^k/q_k} \le m^{md_j^k/q_k},$$

for each $k = 1, \ldots, n$, and so

$$l^{l\rho_j}\sum_{k=1}^{n}(m-l)^{(m-l)d_j^k/q_k} \le \sum_{k=1}^{n} m^{md_j^k/q_k}.$$

Also,

$$\begin{aligned}\sum_{l=0}^{m}\binom{m}{l}(l+\lambda sM)^{m-l} &\le \sum_{l=0}^{m}\binom{m}{l}[1+(l+\lambda sm)]^{m-l}\\ &\le \sum_{l=0}^{m}\binom{m}{l}[1+(m+\lambda sM)]^{m-l} = (m+1+\lambda sM)^m.\end{aligned}$$

Hence

$$\begin{aligned}&\lambda sM\sum_{l=0}^{m}\binom{m}{l}(l+\lambda sM)^{m-l}+1\\ &\quad\le \lambda sM(m+1+\lambda sM)^m+1 \le (m+1+\lambda sM)^{m+1}.\end{aligned}$$

It therefore follows from (21) that

$$\begin{aligned}\left|\frac{\partial^{m+1}u(x)}{\partial x_j^{m+1}}\right| &\le A_0C_1^{m+1}(m+1+\lambda sM)^{m+1}\left(\sum_{k=1}^{n}(m+1)^{(m+1)d_j^k/q_k}\right)\\ &\quad\cdot\exp\left(c\sum_{k=1}^{n}|x_k|^{p_k}\right), \qquad x\in R^n.\end{aligned}$$

This proves, by induction, that for any $j = 1, \ldots, n$, and $m = 1, 2, \ldots,$

$$\begin{aligned}\left|\frac{\partial^{m}u(x)}{\partial x_j^{m}}\right| &\le A_0C_1^{m}(m+\lambda sM)^{m}\left(\sum_{k=1}^{n}m^{md_j^k/q_k}\right)\exp\left(c\sum_{k=1}^{n}|x_k|^{p_k}\right)\\ &\le A_0e^{\lambda sM}C_1^m\left(\sum_{k=1}^{n}m^{m(1+d_j^k/q_k)}\right)\exp\left(c\sum_{k=1}^{n}|x_k|^{p_k}\right), \qquad x\in R^n.\end{aligned}$$

The theorem is thus established with the constants

$$A_1 = A_0e^{\lambda sM}, \qquad C_1 \quad\text{and}\quad c_1 = c.$$

REMARK. If $1 < \max_{1\le k\le n} d_j^k/p_k$, then $1+\max_{1\le k\le n} d_j^k/q_k < \max_{1\le k\le n} d_j^k$, i.e. the regularity of the solution u in the jth variable is improved due to the growth condition (6).

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