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THE H-DEVIATION OF A LIFT

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ABSTRACT. Let X be an H-space, $x \in X$ a primitive element, and α a stable primary operation that vanishes on x. Let y represent x in $H^*(P_2X)$. (P_2X is the projective plane of X.) Let $\tilde{D}$ be the H-deviation of the lift of x to the two-stage Postnikov system with k-invariant α. We obtain a formula that relates $\tilde{D}$ to the representation of $\alpha(y)$ as a product in $H^*(P_2X)$.

We consider the following data: X a CW H-space, $g: K_1 \to K_0$ a homomorphism of topological groups, and $(f, F): X \to \Omega K_1$ an H-map, such that $\Omega g \circ f$ is nullhomotopic. We thus have a diagram

$$
* \qquad
\begin{tikzcd}
 & & \Omega^2 K_0 \arrow[d, "\Omega j"] & \\
 & & \Omega E \arrow[d, "\Omega p"] & \\
X \arrow[r, "f"] \arrow[urr, "f_1"] & \Omega K_1 \arrow[r, "\Omega g"] & \Omega K_0 &
\end{tikzcd}
$$

Here E is the homotopy fiber of g and the lift f_1 is given by

$$f_1(x) = (f(x), l(x)) \in \Omega E \subset \Omega K_1 \times P\Omega K_0,$$

where l is a nullhomotopy $l: * \sim \Omega g \circ f$. Since f is an H-map, the H-deviation Df_1 of f_1 factors as

$$X \wedge X \xrightarrow{\tilde{D}} \Omega^2 K_0 \xrightarrow{\Omega j} \Omega E,$$

where $\tilde{D}$ is a map we shall describe in detail below. In [3], Zabrodsky considered the above situation and proved (Proposition 3.2.2).

THEOREM A$'$. *In the diagram* $(*)$, *suppose that*
(a) X *is a loop space,* $X = \Omega X'$;
(b) f *is a loop map,* $f = \Omega f'$; *and*
(c) $g \circ f'$ *factors as*

$$X' \xrightarrow{\overline{\Delta}} X' \wedge X' \xrightarrow{\theta'} K_0$$

for some map θ', *where* $\overline{\Delta}$ *is the reduced diagonal.*

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If $l = \theta' \circ \xi$, where ξ is the standard nullhomotopy of $\Omega\overline{\Delta}$ then $\tilde{D}$ is adjoint to the composition θ given by

$$\Sigma X \wedge \Sigma X \xrightarrow{\varepsilon \wedge \varepsilon} X' \wedge X' \xrightarrow{\theta'} K_0,$$

where ε is the evaluation map.)

In particular, there is the immediate corollary:

THEOREM B$'$. *Under the hypotheses of Theorem* A$'$, *if K_0 is a generalized Eilenberg-Mac Lane space with field coefficients and*

$$[g \circ f'] = \sum_i (-1)^{\deg(y_i')} y_i' y_i'' \in H^*(X'),$$

then by choosing θ' such that

$$[\theta'] = \sum_i y' \otimes y'' \in H^*(X' \wedge X')$$

we get

$$[\tilde{D}] = \sum_i \sigma^* y_i' \otimes \sigma^* y_i'' \in H^*(X \wedge X).$$

These results have proved extremely useful in the study of H-spaces, but their utility is sometimes restricted due to the fact that hypotheses (a), (b), or (c) of Theorem A$'$ may not hold. In this paper we shall show that a formula for $\tilde{D}$ can be obtained in the absence of these hypotheses.

We establish some notation. Let

$$\Sigma X \xrightarrow{h} P_2 X \xrightarrow{\hat{\Delta}} C_h \sim \Sigma^2 (X \wedge X)$$

define C_h as the mapping cone of the inclusion of the suspension in the projective plane. Let $F: P_2 X \to K_1$ be induced by the H-structure of f. (We shall justify this notation below.)

THEOREM A. *The map $g \circ F$ factors as $\theta \circ \hat{\Delta}$, where $\theta: \Sigma^2 (X \wedge X) \to K_0$ is adjoint to $\tilde{D}$.*

Theorem A has the immediate corollary (using pp. 495–496 of [**1**]):

THEOREM B. *Let K_0 be a generalized Eilenberg-Mac Lane space with field coefficients*

(a) *If $[g \circ F] = \sum_i (-1)^{\deg(z_i')} z_i' z_i'' \in H^*(P_2 X)$, then*

$$[\tilde{D}] = \sum_i h^* z_i' \otimes h^* z_i'' \in H^*(X \wedge X),$$

modulo $\mathrm{Im}(\overline{\mu}^*)$.

(b) *Conversely, if*

$$[\tilde{D}] = \sum_i x_i' \otimes x_i'',$$

then

$$[g \circ F[= \hat{\Delta}^* \Big(\sum_i \otimes x_i' \otimes x_i'' \Big).$$

Hence, if the x_i' *and* x_i'' *are primitive and* $x_i' = h^*(z_i')$, $x_i'' = h^*(z_i'')$, *then*

$$[g \circ F] = \sum_i (-1)^{\deg(z_i')} z_i' z_i''.$$

We remark that the indeterminacy of $[\tilde{D}]$ resulting from different choices of the nullhomotopy l consists of exactly $\mathrm{Im}(\overline{\mu}^*)$. Hence there exists a lift f_1 with precisely the H-deviation given by Theorem B.

Before turning to the proof of Theorem A, let us make explicit the way in which our theorem extends that of Zabrodsky. Under the hypotheses of Theorem A$'$, the evaluation map ε factors as

$$\Sigma X \xrightarrow{h} P_2 X \xrightarrow{k} X',$$

as described on pp. 18–19 of [**2**]. As in Theorem B$'$, write

$$[g \circ f'] = \sum_i (-1)^{\deg(y_i)} y_i' y_i'' \in H^*(X').$$

To apply Theorem B, we set $F = f' \circ k$, so that

$$[g \circ F] = \sum_i (-1)^{\deg(z_i)} z_i' z_i'' \in H^*(P_2 X),$$

where $z_i' = k^*(y_i')$ and $z_i'' = k^*(y_i'')$. By Theorem B,

$$[\tilde{D}] = \sum_i h^* z_i' \otimes h^* z_i'' = \sum_i \sigma^* y_i' \otimes \sigma^* y_i'',$$

modulo $\mathrm{Im}(\overline{\mu})$. Thus, up to the indeterminacy of f_1, Theorem B$'$ follows from Theorem B.

We wish to thank James P. Lin for suggesting the problem considered in this note. We also wish to thank the University of California–San Diego for it's hospitality during the time this work was done. The rest of the paper is devoted to the proof of Theorem A.

Let Δ^n denote the standard n-simplex. We may represent the H-structure of the map $f\colon X \to \Omega K_1$ as

$$F\colon \Delta^2 \times X^2 \to K_1$$

such that

$$\begin{aligned} F(t_0, t_1, 0, x_1, x_2) &= f(x_1)[t_0], \\ F(0, t_1, t_2, x_1, x_2) &= f(x_2)[t_1], \\ F(t_0, 0, t_2, x_1, x_2) &= f(x_1 x_2)[t_0], \\ F(t_0, t_1, t_2, *, x_2) &= f(x_2)[t_0 + t_1], \end{aligned}$$

and

$$F(t_0, t_1, t_2, x_1, *) = f(x_1)[t_0].$$

Checking the construction of $P_2 X$ as a quotient space of $\Delta^2 \times X^2$ [**2**, p. 8] we see that F respects precisely those identifications and hence may be regarded as a map $F\colon P_2 X \to K_1$.

The map $\tilde{D}: X \wedge X \to \Omega^2 K_0$ is given by the formula (cf. [**3**, p. 73]):

$$\tilde{D}(x_1, x_2) = l(x_1 x_2) + P\Omega g \circ F(x_1, x_2) - l(x_1)l(x_2),$$

in which juxtaposition denotes multiplication in X and in ΩK_0 and addition denotes the concatenation of paths. The adjoint of $\tilde{D}$ can be expressed as a map

$$\theta = \dot{\Delta}^3 \times X^2 \to K_0$$

given by the picture:

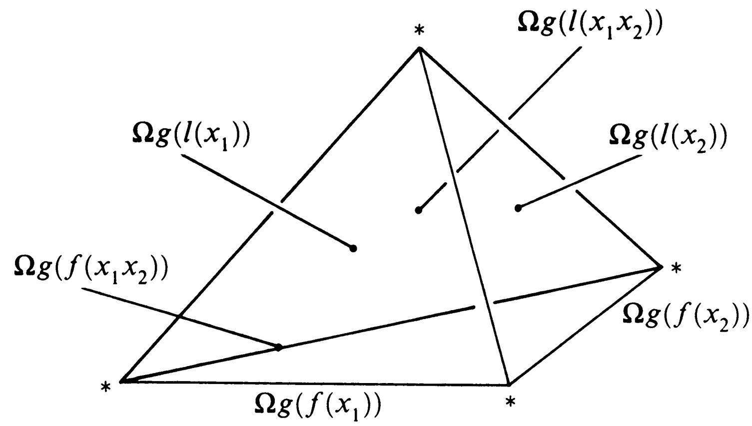

FIGURE 1

Consider the diagram

$$\begin{tikzcd}
 & \Sigma X \arrow[r, "\tilde{f}_1"] \arrow[d, "h"'] & E \arrow[d, "p"] \\
 & P_2X \arrow[r, "F"] \arrow[ur, dashed] \arrow[d, "\bar{\Delta}"'] & K_1 \arrow[d, "g"] \\
P_2X \cup C\Sigma X \arrow[r, phantom, "="] & C_h \arrow[r, "\theta"] & K_0
\end{tikzcd}$$

in which $\tilde{f}_1$ is the adjoint of f_1. The space C_h may be represented as a quotient of $\dot{\Delta}^3 \times X^2$ by coordinates

$$P_2X = \{(t_0, t_1, t_2, 0, x_1, x_2)\}$$

subject to the appropriate identifications, and

$$C\Sigma X = \{(t_0, t_1, t_2, t_3, x_1, x_2) \,|\, t_i = 0, \ i = 0, 1, 2\}$$

with t_3 as the cone parameter. Clearly θ makes the lower square commute. (In fact, from standard obstructioin theory θ is exactly the obstruction to the existence of a diagonal arrow that makes both triangles of the upper square commute.) Theorem A follows.

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