

MIXED-NORM GENERALIZATIONS OF BERGMAN SPACES AND DUALITY

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ABSTRACT. Conditions sufficient for boundedness of the Bergman projection on certain "mixed-norm" spaces of functions on the unit ball of \mathbf{C}^N are given, and this is used to identify the dual space of such mixed-norm spaces. Several representation theorems that follow from the duality are also given.

1. Introduction. The classical Bergman space A^p on the unit ball $B = B_N$ in \mathbf{C}^N is the set of functions $f \in H(B_N)$ satisfying

$$\left(\int_B |f(z)|^p dm(z) \right)^{1/p} < \infty.$$

Here $0 < p < \infty$, the space \mathbf{C}^N is equipped with the usual inner product defined for $z = (z_1, \dots, z_N)$ and $w = (w_1, \dots, w_N)$ in \mathbf{C}^N by $\langle z, w \rangle = \sum_{i=1}^N z_i \bar{w}_i$ and with the associated norm $|z| = \langle z, z \rangle^{1/2}$, $H(B_N)$ is the set of holomorphic functions on B_N , and m is Lebesgue measure on B_N normalized so that $m(B_N) = 1$. Using "polar coordinates" (see [6, 1.4.3]), this integral may be written as

$$\left(2N \int_I \left(\int_S |f(r\tau)|^p d\sigma(\tau) \right) r^{2N-1} dr \right)^{1/p}$$

where $I = [0, 1)$, $S = S_N = \partial B_N$, and σ is the rotation invariant positive Borel measure on S_N with $\sigma(S_N) = 1$.

We study here the following weighted "mixed norm" generalizations of the Bergman spaces. If $0 < p, q < \infty$ and if $\alpha > -1$, define

$$A_{\alpha}^{p,q} = \left\{ f \in H(B) \mid \|f\|_{p,q,\alpha} < \infty \right\},$$

where

$$\|f\|_{p,q,\alpha} \equiv \left(\int_I \left(\int_S |f(r\tau)|^p d\sigma(\tau) \right)^{q/p} (1-r^2)^{\alpha} r^{2N-1} dr \right)^{1/q}.$$

Note that when $q = p$ and $\alpha = 0$, this is precisely the Bergman space. Also note that

$$\|f\|_{p,q,\alpha} = \left(\int_I \|f_r\|_{L^p(S)}^q (1-r^2)^{\alpha} r^{2N-1} dr \right)^{1/q},$$

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where, for $0 \leq r < 1$, f_r is the function defined on S by $f_r(\tau) = f(r\tau)$. Using this notation, we also define

$$\begin{aligned} \|f\|_{\infty, q, \alpha} &\equiv \left(\int_I \|f_r\|_{L^\infty(S)}^q (1-r^2)^\alpha r^{2N-1} dr \right)^{1/q}, \\ \|f\|_{p, \infty} &\equiv \sup_{0 \leq r < 1} \|f_r\|_{L^p(S)}, \quad \text{and} \\ \|f\|_{\infty, \infty} &\equiv \sup_{0 \leq r < 1} \|f_r\|_{L^\infty(S)}, \end{aligned}$$

and denote the spaces of holomorphic functions for which these are finite by $A_\alpha^{\infty q}$, $A^{p\infty}$, and $A^{\infty\infty}$ respectively. (The space $A^{p\infty}$ is the classical Hardy space H^p .) The set of (equivalence classes of) measurable functions satisfying the integrability condition defining A_α^{pq} is denoted by L_α^{pq} . It is easy to check that

$$\|f + g\|_{p, q, \alpha}^s \leq \|f\|_{p, q, \alpha}^s + \|g\|_{p, q, \alpha}^s$$

where $s = \min\{p, q, 1\}$; thus $(A_\alpha^{pq}, \|\cdot\|_{p, q, \alpha}^s)$ and $(L_\alpha^{pq}, \|\cdot\|_{p, q, \alpha}^s)$ are metric spaces, and are normed linear spaces if $1 \leq p, q \leq \infty$. The basic reference for mixed norm spaces (including, but not limited to, our spaces L_α^{pq}) is Benedek and Panzone, [2].

Note that by two applications of Hölder's inequality, $L_{\alpha_1}^{p_1 q_1} \subset L_{\alpha_2}^{p_2 q_2}$ if either

$$p_2 \leq p_1 \quad \text{and} \quad q_2 < q_1 \quad \text{and} \quad \frac{(\alpha_1 + 1)}{q_1} < \frac{(\alpha_2 + 1)}{q_2}$$

or

$$p_2 \leq p_1 \quad \text{and} \quad q_2 = q_1 \quad \text{and} \quad \alpha_1 \leq \alpha_2.$$

In either case, the containment is proper (unless $p_1 = p_2$, $q_1 = q_2$, and $\alpha_1 = \alpha_2$), since then there is some constant s satisfying the inequality $N/p_1 + (\alpha_1 + 1)/q_1 < s < N/p_2 + (\alpha_2 + 1)/q_2$, and $f(z) = (1 - \langle z, \zeta \rangle)^{-s}$ (with $\zeta \in S$ fixed) defines a function in $A_{\alpha_2}^{p_2 q_2}$ not in $A_{\alpha_1}^{p_1 q_1}$:

$$\begin{aligned} \|f\|_{p, q, \alpha}^q &= \int_I \left(\int_S |1 - \langle r\tau, \zeta \rangle|^{-sp} d\sigma(\tau) \right)^{q/p} (1-r^2)^\alpha r^{2N-1} dr \\ &\sim \int_I \left((1-r)^{N-sp} \right)^{q/p} (1-r)^\alpha r^{2N-1} dr \end{aligned}$$

by Proposition 1.4.10 of [6], and this integral is finite if and only if $s < N/p + (\alpha + 1)/q$. (We write $a(x) \sim b(x)$ if there exist constants c and C such that $0 < c < a(x)/b(x) < C < \infty$ as x ranges over some index set.)

A few of the basic properties of A_α^{pq} are put forth in section two. Section three deals with boundedness of the Bergman projection, and this result is used to identify the dual space of our mixed norm spaces in §4. After introducing the pseudohyperbolic metric in §5, §6 and §7 are concerned with representations of the mixed norms and of functions in the mixed norm spaces. These results are generalizations of work of Luecking in [4].

This work represents some of the results contained in the author's Ph.D. dissertation completed at Michigan State University under the direction of Professor William Sledd.

2. Basic properties of A_α^{pq} . The completeness of A_α^{pq} is a consequence of the following growth condition. Our statements and proofs here will resemble those in [7].

PROPOSITION 2.1. *If $f \in A_\alpha^{pq}$ ($0 < p, q \leq \infty, -1 < \alpha$), then*

$$|f(z)| \leq C \|f\|_{p,q,\alpha} (1 - |z|)^{-(N/p + (\alpha+1)/q)} \text{ for every } z \in B$$

for some C independent of f .

PROOF. First suppose $0 < p < \infty$. Since $|f|^p$ is plurisubharmonic, for $0 < r < s < 1$ and $\tau \in S$ we have

$$\begin{aligned} |f(r\tau)|^p &\leq \int_S |f(s\eta)|^p P((r/s)\tau, \eta) d\sigma(\eta) \\ &\leq 2^N (s-r)^{-N} \int_S |f(s\eta)|^p d\sigma(\eta), \end{aligned}$$

i.e.,

$$(2.1) \quad |f(r\tau)|(s-r)^{N/p} \leq C \|f_s\|_{L^p(S)}.$$

Here P denotes the invariant Poisson kernel defined for $z \in B$ and $\zeta \in S$ by

$$P(z, \zeta) = \left(\frac{1 - |z|^2}{|1 - \langle z, \zeta \rangle|^2} \right)^N.$$

For basic facts concerning the invariant Poisson kernel, see [6, Section 3.3]. If $q = \infty$, the result follows from (2.1) immediately. If $0 < q < \infty$, we then have

$$|f(r\tau)|^q \int_r^1 (s-r)^{Nq/p} (1-s)^\alpha s^{2N-1} ds \leq C \|f\|_{p,q,\alpha}^q.$$

Letting $x = (s-r)/(1-r)$, for $1/2 \leq r < 1$ it follows that

$$\begin{aligned} C \|f\|_{p,q,\alpha}^q &\geq |f(r\tau)|^q (1-r)^{(N/p + (\alpha+1)/q)q} \int_0^1 x^{Nq/p} (1-x)^\alpha [(1-r)x + r]^{2N-1} dx \\ &\sim |f(r\tau)|^q (1-r)^{(N/p + (\alpha+1)/q)q} \int_0^1 x^{Nq/p} (1-x)^\alpha dx \end{aligned}$$

so the result follows. If $0 \leq r < 1/2$, the result follows from the maximum modulus theorem.

Now suppose $p = \infty$. Then $|f(r\tau)| \leq \|f_s\|_{L^\infty(S)}$ for $0 \leq r \leq s < 1$, and the result for $0 < q \leq \infty$ is proven by the same procedures.

COROLLARY 2.2. *If $1 \leq p, q \leq \infty$ and $-1 < \alpha$, then A_α^{pq} is a closed subspace of L_α^{pq} , and is hence a Banach space.*

PROOF. Suppose $f_n \rightarrow f$ in L_α^{pq} with $f_n \in A_\alpha^{pq}$. Then there is a subsequence f_{n_k} with $f_{n_k} \rightarrow f$ pointwise a.e. [2, p. 304]. By Proposition 2.1, we also have that f_n is uniformly Cauchy on compact subsets of B , so f_n is uniformly convergent on compact subsets of B to some g since L_α^{pq} is complete [2, p. 304]. But g is analytic [6, 1.1.4] and $f = g$ a.e., so $g \in A_\alpha^{pq}$ and $f_n \rightarrow g$ in A_α^{pq} .

PROPOSITION 2.3. *If $0 < p, q < \infty$, then $\lim_{r \rightarrow 1^-} \|f_r - f\|_{p,q,\alpha} = 0$ for every $f \in A_\alpha^{pq}$.*

This follows immediately from the dominated convergence theorem. (For details, see [7, Proposition 3.3].) So the functions analytic in a neighborhood of B form a dense subset of A_α^{pq} .

3. Boundedness of the Bergman projection on L_α^{pq} . Suppose $t > -1$. The Bergman kernel K_t is defined by

$$K_t(z, w) = \frac{(1 - |w|^2)^t}{(1 - \langle z, w \rangle)^{N+1+t}}$$

for $z, w \in B_N$. Note that:

- (a) for fixed $w \in B$, $K_t(\cdot, w) \in A_\alpha^{pq}$ and
- (b) for fixed $z \in B$, $K_t(z, \cdot) \in L_\alpha^{pq}$ if $tq > -(\alpha+1)$. (But $K_t(z, \cdot)$ is not conjugate holomorphic unless $t = 0$.)

Both observations follow because the denominator of K_t is bounded above and below in B .

The Bergman projection T_t is defined by

$$T_t f(z) = \binom{N+t}{N} \int_B K_t(z, w) f(w) dm(w)$$

for $z \in B_N$ and f for which the integrands are in $L^1(dm)$. In general, the binomial coefficient $\binom{N+t}{N}$ is $\frac{\Gamma(N+t+1)}{\Gamma(N+1)\Gamma(t+1)}$. It is clear that, for fixed t , $T_t f$ is holomorphic when defined.

In this section, a condition on t, p, q , and α will be found which ensures that T_t is bounded on L_α^{pq} ; there will be no dependence on p other than $p \geq 1$. In [3], Forelli and Rudin showed that T_t is bounded on $L^p(dm)$ ($1 \leq p < \infty$) if and only if $(t+1)p > 1$. Then in [1] Békollé showed that T_t is bounded on $L^p(dm_\alpha)$ ($1 < p < \infty, -1 < \alpha, dm_\alpha(z) = (1 - |z|^2)^\alpha dm(z)$) if and only if $(t+1)p > \alpha + 1$. (He actually showed this for more general weights satisfying a “ B_p condition”, a condition analogous to Muckenhoupt’s A_p condition introduced in [5].) An important tool here will be the following pair of facts due to Forelli and Rudin in [3, Proposition 2.7]:

$$(3.1a) \quad \int_B |K_t(z, w)|(1 - |w|^2)^{-c} dm(w) \leq C(1 - |z|^2)^{-c}$$

for every $z \in B$ if $0 < c < t + 1$;

$$(3.1b) \quad \int_B |K_t(z, w)|(1 - |z|^2)^{-c} dm(z) \leq C(1 - |w|^2)^{-c}$$

for every $w \in B$ if $0 < c + t < t + 1$.

THEOREM 3.1. T_t maps L_α^{pq} ($1 \leq p \leq \infty, 1 < q < \infty, -1 < \alpha$) boundedly into A_α^{pq} if $(t+1)q > \alpha + 1$. Furthermore, $T_t f = f$ and $T_t \bar{f} \equiv \overline{f(0)}$ for every $f \in A_\alpha^{pq}$.

PROOF. As noted in [6, Proposition 7.1.2], $T_t f = f$ and $T_t \bar{f} \equiv \overline{f(0)}$ are true for $f \in H^\infty(B)$, hence for $f \in A_\alpha^{pq}$ by density of $H^\infty(B)$ in A_α^{pq} , once continuity is verified.

If $1 < p < \infty$, vector-valued integration and Hölder's inequality yield that

$$\begin{aligned} \|(T_t f)_r\|_{L^p(S)} &= C \left\| \int_I \int_S K_t(r, s\eta) f_s(\eta) d\sigma(\eta) s^{2N-1} ds \right\|_{L^p(S)} \\ &\leq C \int_I \left\| \int_S K_t(r, s\eta) f_s(\eta) d\sigma(\eta) \right\|_{L^p(S)} s^{2N-1} ds \\ &= C \int_I \left(\int_S \left| \int_S K_t(r\tau, s\eta) f_s(\eta) d\sigma(\eta) \right|^p d\sigma(\tau) \right)^{1/p} s^{2N-1} ds \\ &\leq C \int_I \left[\int_S \left(\int_S |K_t(r\tau, s\eta)| |f_s(\eta)|^p d\sigma(\eta) \right) \right. \\ &\quad \left. \times \left(\int_S |K_t(r\tau, s\eta)| d\sigma(\eta) \right)^{p/p'} d\sigma(\tau) \right]^{1/p} s^{2N-1} ds. \end{aligned}$$

(As usual, p' denotes the conjugate exponent of p .) But $\int_S |K_t(r\tau, s\eta)| d\sigma(\eta)$ is independent of τ , so by Fubini's theorem, the expression above is less than

$$\begin{aligned} C \int_I \left[\left(\int_S |K_t(r\tau, s\eta)| d\sigma(\eta) \right)^{p/p'+1} \left(\int_S |f_s(\eta)|^p d\sigma(\eta) \right) \right]^{1/p} s^{2N-1} ds \\ = C \int_I \int_S |K_t(r\tau, s\eta)| d\sigma(\eta) \|f_s\|_{L^p(S)} s^{2N-1} ds. \end{aligned}$$

This estimate can also be verified in similar way if $p = 1$ or $p = \infty$.

Using this estimate and Hölder's inequality, we have

$$\begin{aligned} \|T_t f\|_{p,q,\alpha}^q &= C \int_I \|(T_t f)_r\|_{L^p(S)}^q (1-r^2)^\alpha r^{2N-1} dr \\ &\leq C \int_I \left[\int_I \int_S |K_t(r\tau, s\eta)| d\sigma(\eta) \|f_s\|_{L^p(S)} s^{2N-1} ds \right]^q (1-r^2)^\alpha r^{2N-1} dr \\ &\leq C \int_I \left(\int_B |K_t(r\tau, s\eta)| (1-s^2)^{-\delta q'} dm(s\eta) \right)^{q/q'} \\ &\quad \times \left(\int_B |K_t(r\tau, s\eta)| (1-s^2)^{\delta q} \|f_s\|_{L^p(S)}^q dm(s\eta) \right) (1-r^2)^\alpha r^{2N-1} dr, \end{aligned}$$

where δ will be chosen later. But

$$\int_B |K_t(r\tau, s\eta)| (1-s^2)^{-\delta q'} dm(s\eta) \leq C(1-r^2)^{-\delta q'}$$

by (3.1a), as long as $0 < \delta q' < t + 1$. Using this and Fubini's theorem, we have

$$\begin{aligned} \|T_t f\|_{p,q,\alpha}^q &\leq C \int_I \left((1-r^2)^{-\delta q'} \right)^{q/q'} \\ &\quad \times \left(\int_B |K_t(r\tau, s\eta)| (1-s^2)^{\delta q} \|f_s\|_{L^p(S)}^q dm(s\eta) \right) (1-r^2)^\alpha r^{2N-1} dr \\ &= C \int_I \|f_s\|_{L^p(S)}^q (1-s^2)^{\delta q} \\ &\quad \times \int_I \int_S |K_t(r\tau, s\eta)| d\sigma(\eta) (1-r^2)^{\alpha-\delta q} r^{2N-1} dr s^{2N-1} ds. \end{aligned}$$

But

$$\begin{aligned} & \int_I \int_S |K_t(r\tau, s\eta)| d\sigma(\eta) (1-r^2)^{\alpha-\delta q} r^{2N-1} dr \\ &= \int_B |K_t(r\tau, s\eta)| (1-r^2)^{-(-\alpha+\delta q)} dm(r\tau) \\ &\leq C(1-s^2)^{-(-\alpha+\delta q)} = C(1-s^2)^{\alpha-\delta q} \end{aligned}$$

by (3.1b), as long as $0 < -\alpha + \delta q + t < t + 1$. So

$$\|T_t f\|_{p,q,\alpha}^q \leq C \int_I \|f_s\|_{L^p(S)}^q (1-s^2)^\alpha s^{2N-1} ds = C \|f\|_{p,q,\alpha}^q.$$

To choose suitable t , note that there exists δ satisfying $0 < \delta q' < t + 1$ and $0 < -\alpha + \delta q + t < t + 1$ if and only if

$$(t + 1)q > \alpha + 1.$$

As in [3, p. 594], we immediately get the following.

COROLLARY 3.2. *For $1 \leq p < \infty$, $1 < q < \infty$, and $-1 < \alpha$, we have $\|f\|_{p,q,\alpha} \leq C \|\operatorname{Re} f\|_{p,q,\alpha}$ for every $f \in H(B)$ with $f(0) = 0$.*

PROOF. Choose $t > (\alpha + 1)/q - 1$. Let $u = \operatorname{Re} f$ and fix $0 < r, s < 1$. Then $f_s = T_t(f_s) = T_t(f_s + \overline{f_s}) = 2T_t(u_s)$, so

$$\int_S |f_s(r\tau)|^p d\sigma(\tau) = 2^p \int_S |T_t u_s(r\tau)|^p d\sigma(\tau).$$

Thus

$$\begin{aligned} \|f_s\|_{p,q,\alpha}^q &= \int_I \left(\int_S |f_s(r\tau)|^p d\sigma(\tau) \right)^{q/p} (1-r^2)^\alpha r^{2N-1} dr \\ &= 2^q \int_I \left(\int_S |T_t u_s(r\tau)|^p d\sigma(\tau) \right)^{q/p} (1-r^2)^\alpha r^{2N-1} dr \\ &= 2^q \|T_t u_s\|_{p,q,\alpha}^q \leq 2^q A^q \|u_s\|_{p,q,\alpha}^q, \end{aligned}$$

and the result follows upon letting $s \rightarrow 1$.

4. Representation of the dual space of A_α^{pq} . Representation of the dual space of A_α^{pq} will follow from boundedness of the Bergman projection (Theorem 3.1). The case $N = 1, \alpha = 0$ was handled by Shapiro in [7, Corollary 3.6]. In [4, Theorem 2.1], Luecking identified the dual space of A_α^p using the boundedness of the Bergman projection on A_α^p and $A_{\alpha(1-p')}^{p'}$.

Given any fixed $g \in A_{\alpha(1-q')}^{p'q'}$, define the linear functional L_g on A_α^{pq} by

$$L_g f = \langle f, g \rangle = \int_B f \overline{g} dm$$

for $f \in A_\alpha^{pq}$. By two applications of Hölder's inequality,

$$|\langle f, g \rangle| \leq \|f\|_{p,q,\alpha} \|g\|_{p',q',\alpha(1-q')}.$$

THEOREM 4.1. *Suppose that $1 < p < \infty$, $\max\{1, \alpha + 1\} < q < \infty$, and $-1 < \alpha$. Then the map taking g to L_g is a linear homeomorphism of $A_{\alpha(1-q')}^{p'q'}$ onto the dual space of A_{α}^{pq} .*

PROOF. As noted above, any $g \in A_{\alpha(1-q')}^{p'q'}$ defines a bounded linear functional L_g on A_{α}^{pq} , with $\|L_g\| \leq \|g\|_{p',q',\alpha(1-q')}$.

Now take any L in the dual space $(A_{\alpha}^{pq})^*$. Extend to $L \in (L_{\alpha}^{pq})^*$ by the Hahn-Banach theorem. Write $\omega(r) = (1 - r^2)^{\alpha}$, and note that $j \in L_{\alpha}^{pq}$ if and only if $j\omega^{-1/q} \in L_{\alpha}^{pq}$. Define the functional $\Lambda \in (L_{\alpha}^{pq})^*$ by $\Lambda j = L(j\omega^{-1/q})$; then there exists some function $k \in L^{p'q'}$ such that $\Lambda j = \int_B jk \, dm$. (See [2, Theorem 3.1] for the generalized representation theorem.) Let $h = \overline{k\omega^{1/q}}$. We have $h \in L_{\alpha(1-q')}^{p'q'}$ and $Lf = \langle f, h \rangle$ for all $f \in L_{\alpha}^{pq}$. Since $\overline{K_0(z, w)} = K_0(w, z)$, Fubini's theorem implies that

$$(4.1) \quad \langle T_0 f_1, f_2 \rangle = \langle f_1, T_0 f_2 \rangle \quad \text{for } f_1 \in L_{\alpha}^{pq} \cap L^2 \text{ and } f_2 \in L_{\alpha(1-q')}^{p'q'} \cap L^2.$$

(To justify the application of Fubini's theorem, note that $T_0 f_1$ and $T_0 f_2$ are in L^2 since f_1 and f_2 are, either by Theorem 3.1 or by Békollé's result.) Now, $q > \alpha + 1$ and $q' > \alpha(1 - q') + 1$, so by Theorem 3.1, T_0 is bounded on L_{α}^{pq} and $L_{\alpha(1-q')}^{p'q'}$. By continuity of T_0 and density of the respective spaces [2, p. 308], (4.1) is also true for $f_1 \in L_{\alpha}^{pq}$ and $f_2 \in L_{\alpha(1-q')}^{p'q'}$. Let $g = T_0 h$. So $g \in A_{\alpha(1-q')}^{p'q'}$, and for $f \in A_{\alpha}^{pq}$ we have $Lf = \langle f, h \rangle = \langle T_0 f, h \rangle = \langle f, T_0 h \rangle = \langle f, g \rangle$, i.e., $L = L_g$.

If $g \in A_{\alpha(1-q')}^{p'q'}$ defines the zero functional, then since $\overline{K_0(z, \cdot)} \in A_{\alpha}^{pq}$ for any fixed $z \in B$, we have $0 = \langle \overline{K_0(z, \cdot)}, g \rangle = \overline{T_0 g(z)} = \overline{g(z)}$, i.e., $g \equiv 0$. So the map taking g to L_g is a one-to-one, continuous, linear transformation of $A_{\alpha(1-q')}^{p'q'}$ onto $(A_{\alpha}^{pq})^*$. By the open mapping theorem, the map is actually a linear homeomorphism.

One can use other (i.e., weighted) duality pairings (and other kernels) to get other representations of $(A_{\alpha}^{pq})^*$.

5. The pseudohyperbolic "metric". The pseudohyperbolic "metric" ρ is defined on B_N by $\rho(z, w) = |\Phi_w(z)|$ where Φ_w is the automorphism of B_N given for $w \neq 0$ by

$$\Phi_w(z) = \frac{w - \frac{\langle z, w \rangle}{\langle w, w \rangle} w - (1 - |w|^2)^{1/2} (z - \frac{\langle z, w \rangle}{\langle w, w \rangle} w)}{1 - \langle z, w \rangle}$$

and for $w = 0$ by $\Phi_0(z) = -z$. The corresponding "balls" are

$$E(w, \delta) = \Phi_w^{-1}(\delta B_N) = \{z \in B_N \mid \rho(z, w) < \delta\}$$

for $w \in B_N$ and $0 < \delta < 1$. Note that $m(E(w, \delta)) \sim \delta^{2N} (1 - |w|)^{N+1}$; see [6, 2.2.7].

We will have need of the following.

LEMMA 5.1. *Fix $0 < r < 1$ and $0 < \delta$ small. Then*

$$\frac{r - \delta}{1 - r\delta} < |z| < \frac{r + \delta}{1 + r\delta} \quad \text{for every } z \in E(r, \delta).$$

(Here $E(r, \delta)$ means $E(w, \delta)$ with $w = (r, 0, \dots, 0) \in B_N$.)

PROOF. Write $z = (z_1, z_2, \dots, z_N) = (z_1, z')$ and suppose $z \in E(r, \delta)$. Then

$$\delta^2 |1 - rz_1|^2 > |r - z_1|^2 + (1 - r^2) |z'|^2,$$

i.e.,

$$2r(1 - \delta^2) \operatorname{Re}(z_1) > (r^2 - \delta^2) + (1 - \delta^2 r^2) |z_1|^2 + (1 - r^2) |z'|.$$

Now, $4r \operatorname{Re}(z_1) + r^2 |z'|^2 < r^2 + 4r |z_1| - r^2 |z_1|^2 \leq 4r < 4$, so

$$\begin{aligned} & (r^2 - \delta^2) + (1 - \delta^2 r^2) |z|^2 \\ &= [(r^2 - \delta^2) + (1 - \delta^2 r^2) |z_1|^2 + (1 - r^2) |z'|^2] + r^2 (1 - \delta^2) |z'|^2 \\ &< 2r(1 - \delta^2) (\operatorname{Re}(z_1) + r |z'|^2 / 2) \\ &\leq 2r(1 - \delta^2) (|z_1|^2 + r |z'|^2 \operatorname{Re}(z_1) + r^2 |z'|^4 / 4)^{1/2} \\ &< 2r(1 - \delta^2) (|z_1|^2 + |z'|^2)^{1/2} = 2r(1 - \delta^2) |z|. \end{aligned}$$

Hence $(|z|, 0') \in E(r, \delta)$, and $(r - \delta)/(1 - r\delta) < |z| < (r + \delta)/(1 + r\delta)$.

6. A norm-representation theorem. In [4, Theorem 5.1], Luecking shows that

$$\|f\|_{H^p} \sim \left(\sup_m \sum_{k=1}^{k_m} |f(a_{mk})|^p (1 - r_m)^N \right)^{1/p} \quad \text{for all } f \in H^p$$

where $0 \leq r_0 < r_1 < \dots \rightarrow 1$ and $\{a_{mk}\}$ satisfies

- (1) $|a_{mk}| = r_m$ for each $m = 0, 1, 2, \dots$ and each $k = 1, 2, \dots, k_m$,
- (2) $r_m S_N \subset \bigcup_k E(a_{mk}, \delta)$ for each m for some $\delta = \delta(p)$ sufficiently small, and
- (3) $E(a_{mk}, \varepsilon) \cap E(a_{mk'}, \varepsilon) = \emptyset$ for each m and each $k \neq k'$ for some $0 < \varepsilon(p) < \delta$.

Such a set of points $\{a_{mk}\}$ will be called an ε - δ lattice.

A close analysis of Luecking's proof yields the following.

THEOREM 6.1. Fix $0 < p < \infty$, $0 < q < \infty$, and $-1 < \alpha$. Let $r_m = 1 - 2^{-m}$ for $m = 0, 1, 2, \dots$ and suppose $\{a_{mk}\}$ is an ε - δ lattice for $\delta = \delta(p, q, \alpha)$ sufficiently small. Then

$$\|f\|_{p,q,\alpha} \sim \left(\sum_{m=0}^{\infty} \left(\sum_{k=1}^{k_m} |f(a_{mk})|^p 2^{-mN} \right)^{q/p} 2^{-m(\alpha+1)} \right)^{1/q} \quad \text{for every } f \in A_\alpha^{pq}.$$

PROOF. Let $r_{m\varepsilon} = (r_m + \varepsilon)/(1 + r_m\varepsilon)$, $E_{mk} = E(a_{mk}, \varepsilon)$, $A_m = \bigcup_k E_{mk}$, and $I_{m\varepsilon} = [(r_m - \varepsilon)/(1 - r_m\varepsilon), (r_m + \varepsilon)/(1 + r_m\varepsilon)]$. Then by plurisubharmonicity of $|f|^p$, the separation property (3), and Lemma 5.1,

$$\begin{aligned} \sum_k |f(a_{mk})|^p 2^{-mN} &\leq 2^{-mN} \left(\sum_k C \frac{\int_{E_{mk}} |f|^p dm}{m(E_{mk})} \right) \\ &\leq C 2^{-mN} 2^{m(N+1)} \sum_k \int_{E_{mk}} |f|^p dm = C 2^m \int_{A_m} |f|^p dm \\ &\leq C 2^m \int_{I_{m\varepsilon}} \left(\int_S |f(r\tau)|^p d\sigma(\tau) \right) r^{2N-1} dr \\ &\leq C 2^m \left(\int_{I_{m\varepsilon}} r^{2N-1} dr \right) \|f_{r_{m\varepsilon}}\|_{L^p(S)}^p \leq C \|f_{r_{m\varepsilon}}\|_{L^p(S)}^p. \end{aligned}$$

Since we may assume that $\varepsilon < 1/3$ (at the cost of increasing the constant C), we have $r_{m\varepsilon} < (r_m + 1)/2 = r_{m+1}$, and thus

$$\sum_m \left(\sum_k |f(a_{mk})|^p 2^{-mN} \right)^{q/p} 2^{-m(\alpha+1)} \leq C \sum_m \|f_{r_{m+1}}\|_{L^p(S)}^q 2^{-m(\alpha+1)} \\ \leq C \int_I \|f_r\|_{L^p(S)}^q (1-r)^\alpha r^{2N-1} dr = C \|f\|_{p,q,\alpha}^q.$$

In the other direction, Luecking uses a change of variables, Fubini's theorem, and the "denseness" property (2) and actually shows that

$$c \|f_{r_m}\|_{L^p(S)}^p \leq C \delta^p \|f_{r_{m\varepsilon}}\|_{L^p(S)}^p + \sum_k |f(a_{mk})|^p 2^{-mN}$$

so

$$c \|f\|_{p,q,\alpha}^q \leq \sum_m \left(C \delta^p \|f_{r_{m+1}}\|_{L^p(S)}^p + \sum_k |f(a_{mk})|^p 2^{-mN} \right)^{q/p} 2^{-m(\alpha+1)} \\ \leq C' \sum_m \left(C \delta^q \|f_{r_{m+1}}\|_{L^p(S)}^q + \left(\sum_k |f(a_{mk})|^p 2^{-mN} \right)^{q/p} \right) 2^{-m(\alpha+1)}$$

where $C' = 1$ if $q \leq p$ and $C' = 2^{q/p-1}$ if $p < q$. This is less than

$$C \delta^q \sum_m \|f_{r_{m+1}}\|_{L^p(S)}^q 2^{-m(\alpha+1)} + C \sum_m \left(\sum_k |f(a_{mk})|^p 2^{-mN} \right)^{q/p} 2^{-m(\alpha+1)} \\ \leq C \delta^q \|f\|_{p,q,\alpha}^q + C \sum_m \left(\sum_k |f(a_{mk})|^p 2^{-mN} \right)^{q/p} 2^{-m(\alpha+1)}$$

and the result follows for δ sufficiently small.

As a consequence of this theorem, no such ε - δ lattice can be a subset of the zero set of an A_α^{pq} function not identically zero.

7. Representation of A_α^{pq} functions. The duality result and the equivalence of norms result (Theorems 3.1 and 6.1) can be used to obtain a representation of A_α^{pq} functions as sums of kernel functions. This generalizes Luecking's Corollary 4.4 in [4].

If v is a weight function on $\{0, 1, 2, \dots\}$, we write $c = \{c_{mk}\}_{m,k} \in l_v^{pq}$ if

$$\left(\sum_{m=0}^\infty \left(\sum_{k=1}^\infty |c_{mk}|^p \right)^{q/p} v_m \right)^{1/q} \equiv \|c\|_{p,q,v} < \infty.$$

THEOREM 7.1. *Suppose $1 < p < \infty$, $1 < q < \infty$, and $-1 < \alpha$. Let $r_m = 1 - 2^{-m}$ and $v_m = (1 - r_m)^{1+Nq/p'+\alpha(1-q')} r_m^{2N-1}$ for each $m = 0, 1, 2, \dots$, and suppose $\{a_{mk}\}$ is an ε - δ lattice for $\delta = \delta(p, q, \alpha)$ sufficiently small. Then every $f \in A_\alpha^{pq}$ is of the form*

$$f(z) = \sum_{m=0}^\infty \sum_{k=1}^{k_m} c_{mk} v_m (1 - \langle z, a_{mk} \rangle)^{-N-1}$$

for some $c \in l_v^{pq}$, and any f of this form is in A_α^{pq} .

Note that no claim of uniqueness of $c \in l_v^{pq}$ is being made.

PROOF. As in the proof of Theorem 6.1,

$$\|g\|_{p',q',\alpha(1-q')} \sim \left(\sum_m \left(\sum_k |g(a_{mk})|^{p'} \right)^{q'/p'} (1-r_m)^{1+Nq'/p'+\alpha(1-q')} r_m^{2N-1} \right)^{1/q'}$$

i.e.,

$$\|g\|_{p',q',\alpha(1-q')} \sim \|g(a_{mk})\|_{p',q',v}$$

Thus the map $R: A_{\alpha(1-q')}^{p'q'} \rightarrow l_v^{p'q'}$ defined by $(Rg)_{mk} = g(a_{mk})$ is a linear isomorphism. Hence R is one-to-one with closed range, and $R^*: l_v^{pq} \rightarrow A_{\alpha}^{pq}$ is onto. (Since $\max\{1, \alpha(1-q') + 1\} < q' < \infty$, we have $(A_{\alpha(1-q')}^{p'q'})^* \sim A_{\alpha}^{pq}$ by Theorem 3.1, where the duality pairing does not involve a weight, and we have $(l_v^{p'q'})^* \sim l_v^{pq}$ by Theorem 3.1 of [2], where the duality pairing does involve the weight v .)

To identify R^* , take $g \in A_{\alpha(1-q')}^{p'q'}$ and $c \in l_v^{pq}$, supposing first that c has only finitely many nonzero terms. Then

$$\begin{aligned} \int_B (R^*c)g \, dm &= \langle R^*c, g \rangle = \langle c, Rg \rangle = \sum_m \left(\sum_k c_{mk} \overline{g(a_{mk})} \right) v_m \\ &= \sum_m \sum_k c_{mk} v_m \overline{\int_B g(z)(1-\langle a_{mk}, z \rangle)^{-N-1} \, dm(z)} \\ &= \int_B \left(\sum_m \sum_k c_{mk} v_m (1-\langle z, a_{mk} \rangle)^{-N-1} \right) \overline{g(z)} \, dm(z), \end{aligned}$$

so $R^*c(z) = \sum_m \sum_k c_{mk} v_m (1-\langle z, a_{mk} \rangle)^{-N-1}$. To get the result for general $c \in l_v^{pq}$, use finite approximations to c (for which the result was just verified), the continuity of R^* , and the fact that convergence in A_{α}^{pq} implies pointwise convergence.

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