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AN INEQUALITY FOR SOME NONNORMAL OPERATORS

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Dedicated to Professor Tsuneo Kanno on his sixtieth birthday with respect and affection

ABSTRACT. An inequality of use in testing convergence of eigenvector calculations is improved. If e_λ is a unit eigenvector corresponding to an eigenvalue λ of a dominant operator A on a Hilbert space H, then

$$|(g, e_\lambda)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda I)g\|^2}$$

for all g in H for which $Ag \neq \lambda g$. The equality holds if and only if the component of g orthogonal to e_λ is also an eigenvector of A. This result is an improvement of Bernstein's result for selfadjoint operators.

1. Statement of the results. An operator A means a bounded linear operator on a complex Hilbert space H. An operator A is called dominant if there is a real number $M_\lambda \geq 1$ such that

$$\|(A-\lambda)^* x\| \leq M_\lambda \|(A-\lambda)x\|$$

for all x in H and for all complex numbers λ. If there is a constant M such that $M_\lambda \leq M$ for all λ, A is called M-hyponormal. The inclusion relation of these classes of nonnormal operators is as follows:

$$\text{Selfadjoint} \subset \text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal}$$
$$\subset M\text{-hyponormal} \subset \text{Dominant},$$

and it is well known that the inclusions above are all proper.

THEOREM 1. *If e_λ is a unit eigenvector corresponding to an eigenvalue λ of a dominant operator A on a Hilbert space H, then*

$$|(g, e_\lambda)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda)g\|^2}$$

for all g in H for which $Ag \neq \lambda g$. The equality holds if and only if the component of g orthogonal to e_λ is also an eigenvector of A. The bound of the right-hand side is $\|(A-\tau)g\|^2/|\lambda - \tau|^2$ for any complex τ.

The corresponding result for selfadjoint operators in Theorem 1 is shown by Bernstein [**1**]. In this note, we extend the Bernstein result to the class of dominant operators, wider than the one of selfadjoint operators, by appropriate modification of Bernstein [**1**] together with the following lemma.

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LEMMA 1. *Let H be a Hilbert space, and A any operator on H. Then*

$$\text{(i)} \qquad \|x\|^2\|Ax\|^2 - |(x, Ax)|^2 = \|x\|^2\|(A-\tau)x\|^2 - |(x,(A-\tau)x)|^2$$

for any vector x in H and any complex τ.

$$\text{(ii)} \qquad \frac{\|x\|^2\|Ax\|^2 - |(x,Ax)|^2}{\|(A-\lambda)x\|^2} \leq \frac{\|(A-\tau)x\|^2}{|\lambda-\tau|^2}$$

for all x in H for which $Ax \neq \lambda x$ and all complex τ and λ.

PROOF OF LEMMA 1. (i) is already shown in [**2**] by straightforward calculation and (ii) is also easily obtained by using (i).

LEMMA 2. *Let H be a Hilbert space, A a dominant operator on H, e_λ a unit eigenvector of A with corresponding eigenvalue λ, and let f be orthogonal to e_λ. Then for any $g = \alpha e_\lambda + f$ with α complex, either $Ag = \lambda g$ or*

$$|\alpha|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g,Ag)|^2}{\|(A-\lambda)g\|^2}.$$

PROOF OF LEMMA 2. First of all, we have $(A-\lambda)g = (A-\lambda)f$ and

$$\begin{aligned}(g,(A-\lambda)g) &= (g,(A-\lambda)f) = (\alpha e_\lambda,(A-\lambda)f) + (f,(A-\lambda)f)\\ &= \alpha((A-\lambda)^* e_\lambda, f) + (f,(A-\lambda)f)\\ &= (f,(A-\lambda)f)\end{aligned}$$

because the hypothesis $(A-\lambda)e_\lambda = 0$ yields $(A-\lambda)^*e_\lambda = 0$ by the definition of dominant operator A. Put $B = A - \lambda$. Since $Bg = Bf$ and $(g,Bg) = (f,Bf)$, by Lemma 1 and the Schwarz inequality, we have

$$\begin{aligned}\frac{\|g\|^2\|Ag\|^2 - |(g,Ag)|^2}{\|(A-\lambda)g\|^2} &= \frac{\|g\|^2\|(A-\lambda)g\|^2 - |(g,(A-\lambda)g)|^2}{\|(A-\lambda)g\|^2}\\ &= \frac{(|\alpha|^2 + \|f\|^2)\|Bf\|^2 - |(f,Bf)|^2}{\|Bf\|^2}\\ &= |\alpha|^2 + \frac{\|f\|^2\|Bf\|^2 - |(f,Bf)|^2}{\|Bf\|^2}\\ &\geq |\alpha|^2.\end{aligned}$$

PROOF OF THEOREM 1. Inequality is shown by Lemma 1 and Lemma 2, and it is also seen that the equality holds if and only if f is an eigenvector of B, equivalently, f is an eigenvector of A in the proof of Lemma 2.

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