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COFINAL FAMILIES OF COMPACTA IN SEPARABLE METRIC SPACES

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ABSTRACT. We show that if X is a Π_1^1-set, then the family of compact subsets of X contains a cofinal (w.r.t. inclusion) subset of cardinality $\mathbf{d}$; the same is true if X is Π_3^1, under strong set-theoretic hypotheses.

All spaces are separable and metrizable. For all undefined notions and unproved assertions, see Engelking [**3**], Kuratowski [**4**], and Moschovakis [**6**]. We work in ZFC, throughout. We assume familiarity with van Douwen's handbook article [**1**]; for this note, however, it will be convenient to let $\mathscr{K}(X)$ be the *space* of all *nonempty* compact subsets of X with the Vietoris topology, rather than the *set* of *all* compacta in X. Van Douwen asks whether $\operatorname{cof}(\mathscr{K}(X)) = k(X) = \mathbf{d}$ if X is analytic (i.e. Σ_1^1), or at least absolutely Borel (i.e. Δ_1^1), and presumably non-σ-compact, having shown (among other things) that always $kc(X) \leq k(X) \leq \operatorname{cof}(\mathscr{K}(X))$, that $kc(X) = \mathbf{d}$ if X is a non-σ-compact and analytic, and that $kc(X) = k(X) = \operatorname{cof}(\mathscr{K}(X)) = \mathbf{d}$ if X is a non-σ-compact absolute $F_{\sigma\delta}$. Our main result is that $\operatorname{cof}(\mathscr{K}(X)) \leq \mathbf{d}$ if X is coanalytic (i.e. Π_1^1); thus, for non-σ-compact absolute Borel sets X, $kc(X) = k(X) = \operatorname{cof}(\mathscr{K}(X)) = \mathbf{d}$. Extensions to sets of higher complexity are also discussed.

PROPOSITION. *For any space X, $\operatorname{cof}(\mathscr{K}(X)) = kc(\mathscr{K}(X))$.*

PROOF. Let $\mathscr{L}$ be cofinal in $\mathscr{K}(X)$. Then each element of $\mathscr{K}(X)$ is contained in some $L \in \mathscr{L}$; so $\mathscr{K}(X) = \bigcup_{L\in\mathscr{L}} \mathscr{K}(L)$, where $\mathscr{K}(L)$ is compact. So $kc(\mathscr{K}(X)) \leq \operatorname{cof}(\mathscr{K}(X))$.

Conversely, let $\mathscr{C}$ be a covering of $\mathscr{K}(X)$ by compact sets. Then for each $C \in \mathscr{C}$, $\bigcup C$ is compact. Let $\mathscr{L} = \{\bigcup C \colon C \in \mathscr{C}\}$; we claim that $\mathscr{L}$ is cofinal in $\mathscr{K}(X)$. Indeed, if $\varnothing \neq K \subseteq X$ is compact, then $K \in C$ for some $C \in \mathscr{C}$; hence, $K \subseteq \bigcup C$, and we are done.

At this point, let us remark that by van Douwen [**1**, Lemma 8.9], for all computations of kc, k and cof, we can restrict ourselves to subsets of the Cantor set 2^ω, since every Π_n^1, Σ_n^1, Δ_n^1 subset of the Hilbert cube is the perfect image of a subset of 2^ω of the same class.

LEMMA. *Let X be a subset of the Cantor set 2^ω. If $n \in \mathbf{N}$, and X is Π_n^1, then so is $\mathscr{K}(X) \subseteq \mathscr{K}(2^\omega) \approx 2^\omega$.*

PROOF. $\{(x, K)\colon x \in K\}$ is closed in $2^\omega \times \mathscr{K}(2^\omega)$. So $\{(x, K)\colon x \in K\} \cap (2^\omega \backslash X) \times \mathscr{K}(2^\omega)$ is Σ_n^1, hence the projection $\{K \in \mathscr{K}(2^\omega)\colon \text{for some } x \in 2^\omega \backslash X, x \in K\}$ of

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this set onto $\mathscr{K}(2^\omega)$ is also Σ^1_n. But the complement of the projected set is just $\{K \in \mathscr{K}(X) : K \cap (2^\omega \backslash X) = \varnothing\} = \mathscr{K}(X)$, so $\mathscr{K}(X)$ is Π^1_n.

THEOREM. *Let X be any Π^1_1-set. Then* $\text{cof}(\mathscr{K}(X)) \leq \mathbf{d}$.

PROOF. By the above remark, let $X \subseteq 2^\omega$. By the lemma, $\mathscr{K}(X)$ is Π^1_1, and by the proposition $\text{cof}(\mathscr{K}(X)) = kc(\mathscr{K}(X))$. By Luzin-Sierpiński [**5**], a Π^1_1-set is the union of $\aleph_1$ Borel sets, $\mathscr{K}(X) = \bigcup_{\alpha<\omega_1} B_\alpha$. Now since B_α is a continuous image of ω^ω, we have $kc(X) \leq kc(\omega^\omega) = \mathbf{d}$; thus $kc(\mathscr{K}(X)) \leq \mathbf{d}$. $\aleph_1 = \mathbf{d}$.

COROLLARY. (a) *Let X be a non-σ-compact absolute Borel set; then* $kc(X) = k(X) = \text{cof}(\mathscr{K}(X)) = \mathbf{d}$.

(b) *Let X be nonlocally compact and coanalytic; then* $\text{cof}(\mathscr{K}(X)) = \mathbf{d}$.

PROOF. (a) follows from the theorem and van Douwen [**1**, Theorem 8.10(e)]; (b) follows from the theorem and van Douwen [**1**, Fact 8.1(c), Lemmas 8.3 and 8.4].

The proof of the theorem shows in fact that $\text{cof}(\mathscr{K}(X)) \leq \mathbf{d}$. κ if $\mathscr{K}(X)$ can be written as a union of κ Borel sets. If X is Π^1_2, then so is $\mathscr{K}(X)$ by the lemma, and it is a theorem of Martin (see Moschovakis [**5**]) that if Σ^1_1 games are determined (and AC holds) then any Σ^1_3-set is a union of $\aleph_2$ Borell sets; if X is Π^1_3, then so is $\mathscr{K}(X)$, and the same theorem of Martin says that if Δ^1_2-games are determined (and AC holds) then any Σ^1_4-set is a union of $\aleph_3$ Borel sets. Furthermore, from a theorem of Steel [**7**] it can easily be deduced (compare van Engelen [**2**, Lemma 4.5.5]) that, if Λ is some Σ^1_n, Π^1_n, or Δ^1_n, then determinancy of Λ games implies that every non-σ-compact set in Λ contains a closed copy of ω^ω. Combining these remarks with van Douwen's results, we have

COROLLARY. (a) *Let X be analytic and non-σ-compact. If analytic games are determined, and $\mathbf{d} \geq \aleph_2$, then* $kc(X) = k(X) = \text{cof}(\mathscr{K}(X)) = \mathbf{d}$.

(b) *Let X be Π^1_2. If Σ^1_1-games are determined, and $\mathbf{d} \geq \aleph_2$, then* $\text{cof}(\mathscr{K}(X)) \leq \mathbf{d}$; *if furthermore X is non-σ-compact and Π^1_2-games are determined, then* $kc(X) = k(X) = \text{cof}(\mathscr{K}(X)) = \mathbf{d}$.

(c) *Let X be Π^1_3. If Δ^1_2-games are determined and $\mathbf{d} \geq \aleph_3$, then* $\text{cof}(\mathscr{K}(X)) \leq \mathbf{d}$; *if furthermore X is non-σ-compact, and Π^1_3-games are determined, then* $kc(X) = k(X) = \text{cof}(\mathscr{K}(X)) = \mathbf{d}$.

Recent results of Jackson combined with well-known results of descriptive set theory imply that if determinancy holds for $L(\mathscr{R})$ (and AC holds) then for each n we can find a natural number $k(n)$ such that each Π^1_n-set is a union of $\aleph_{k(n)}$ Borel sets. Thus

COROLLARY. *If* $\text{Det}(L(\mathscr{R}))$ *and $\mathbf{d} \geq \aleph_{k(n)}$, then for X a non-σ-compact Π^1_n-set,* $kc(X) = k(X) = \text{cof}(\mathscr{K}(X)) = \mathbf{d}$.

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